

JACKSON NETWORKS WITH UNLIMITED SUPPLY OF WORK

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Abstract

We consider a Jackson network in which some of the nodes have an infinite supply of work: When all the customers queued at such a node have departed, the node will process a customer from this supply. Such nodes will be processing jobs all the time, so they will be fully utilized and experience traffic intensity of 1. We calculate flow rates for such networks, obtain conditions for stability, and investigate the stationary distributions: Standard nodes in this network continue to have product form distributions, while nodes with an infinite supply of work have geometric marginal distributions and Poisson inflows and outflows, but their joint distribution is not of product form.

Keywords: queues, Jackson networks, infinite virtual buffers

2000 Mathematics Subject Classification: Primary 60K25,90B22,68M20,90B15
Secondary

Consider a Jackson network [5] with nodes i , service rates μ_i , exogenous input rates α_i , and routing probabilities P_{ij} , $i, j = 1, \dots, I, j \neq i$. Denote by $Q_i(t)$, $t > 0$ the number of items in node i at time t . In addition assume that a subset of the nodes $E \subseteq \{1, \dots, I\}$ have an infinite supply of work, by which we mean the following: For $i \in E$, when $Q_i(t) = 0$, a new item is picked from an infinite supply of items and is processed by the node at the rate μ_i . Upon completion of processing it is routed according to P_{ij} . At each of the nodes $i \in E$ items which arrive at the node have preemptive priority over items from the infinite supply. After its initial processing an item from the infinite supply is treated like any other item.

We believe that such Jackson networks with an infinite supply of work are a useful and realistic model for some situations: Consider a communication network, where each node is transmitting messages originating at this node, with unlimited supply of material to transmit. In addition each node also serves to transmit messages which are in transit between other nodes. Assume that each node gives preemptive priority to messages in transit, over its own messages. When only messages in transit are counted as congestion, this is exactly our system. A particular computer communication system which works in this way is a MAN (metropolitan area networks) Ethernet RPR (resilient packet ring), in which ring traffic has priority over the traffic generated at nodes.

The idea of an infinite supply (or backlog) of lower priority work in a system has been used frequently, e.g [6, 3]. Jackson networks with an infinite supply of work are a special case of multiclass queueing networks with virtual infinite buffers, introduced in [2, 8, 9, 10].

Recall that all service and interarrival times in a Jackson network are independent and exponentially distributed. Assume also that the routing matrix P has spectral

RESEARCH SUPPORTED IN PART BY ISRAEL SCIENCE FOUNDATION GRANT 249/02

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radius < 1 , so that with probability one every item leaves the system after a finite number of processing steps. We denote $\bar{E} = \{i : i \notin E\}$, use λ, μ, α to denote the vectors, and use subscripts $_{E, \bar{E}, E\bar{E}}$, to denote subvectors or submatrices.

Let λ_i denote the rate at which items arrive into node i , counting exogenous input or routing from other nodes. In equilibrium the rate at which items depart from node i is λ_i , $i \in \bar{E}$ and μ_i , $i \in E$. Hence the traffic equations for this system are

$$\lambda_i = \alpha_i + \sum_{j \in \bar{E} \neq i} \lambda_j P_{ji} + \sum_{j \in E \neq i} \mu_j P_{ji}. \quad (1)$$

These equations are solved by:

$$\begin{aligned} \lambda_{\bar{E}} &= (I - P'_{\bar{E}\bar{E}})^{-1} (\alpha_{\bar{E}} + P'_{\bar{E}\bar{E}} \mu_E) \\ \lambda_E &= \alpha_E + P'_{EE} \mu_E + P'_{\bar{E}E} \lambda_{\bar{E}} \\ &= \alpha_E + P'_{\bar{E}E} (I - P'_{\bar{E}\bar{E}})^{-1} \alpha_{\bar{E}} + \left(P'_{EE} + P'_{\bar{E}E} (I - P'_{\bar{E}\bar{E}})^{-1} P'_{\bar{E}E} \right) \mu_E. \end{aligned} \quad (2)$$

Formulae similar to (2) were derived by Goodman and Massey [4], in a paper on transient Jackson networks, in which nodes $i \in E$ were unstable.

A necessary condition for stability is $\mu > \lambda$. The nodes with the infinite supply of work introduce new items into the system at rates:

$$\eta_E = \mu_E - \lambda_E. \quad (3)$$

It is easy to establish that $\mu > \lambda$ is sufficient for stability, and indeed to partially derive steady state distributions.

Proposition 1. *Assume $\rho_i = \frac{\lambda_i}{\mu_i} < 1$, $i = 1, \dots, I$.*

(i) *For nodes $i \in \bar{E}$ the joint steady state distribution is of product form:*

$$\lim_{t \rightarrow \infty} \mathbf{P}\{Q_i(t) = n_i, i \in \bar{E}\} = \prod_{i \in \bar{E}} (1 - \rho_i) (\rho_i)^{n_i}. \quad (4)$$

(ii) *For nodes $i \in E$ the marginal steady state distribution is*

$$\lim_{t \rightarrow \infty} \mathbf{P}\{Q_i(t) = m\} = (1 - \rho_i) (\rho_i)^m, \quad i \in E. \quad (5)$$

(iii) *The departure streams from node $i \in E$ to all other nodes $j \neq i$ are independent Poisson streams of rates $\mu_i P_{ij}$.*

(iv) *The arrival streams into node $i \in E$ from all other nodes $j \neq i$ are independent Poisson streams of rates $\lambda_j P_{ji}$, $j \in \bar{E}$ and $\mu_j P_{ji}$, $j \in E$.*

The key observation for the proof is that each of the nodes $i \in E$ works non-stop, processing items for i.i.d. exponential times at rate μ_i . Hence departures from the nodes with the infinite supply of work consist of independent Poisson streams. Thus the subnetwork of nodes $i \in \bar{E}$ behaves like a Jackson network, with Poisson inputs.

The difference between the nodes with an infinite supply of work and the standard nodes is intriguing: For the nodes with an infinite supply of work all the input and output streams are Poisson. It is well known [7] that the streams of items moving between nodes in the Jackson part of the network, $i \in \bar{E}$ are not necessarily Poisson

(all the streams of items entering and leaving the subnetwork are Poisson, and streams between say node i and node j may be Poisson in some special cases, for example in overtake-free networks. However in general the stream of customers moving from node i to node j in a Jackson network need not be Poisson). On the other hand, the product form, i.e. the independence of the queue length at different nodes when observed all at the same time t , is lost for nodes $i \in E$: It is no longer true that $Q_i(t), Q_j(t), i, j \in E$ in steady state are independent. This loss of independence can be observed in the following example.

In a recent paper Adan and Weiss [1] analyzed a two node Jackson network with an infinite supply of work, and no exogenous input. This system is depicted in Figure 1.

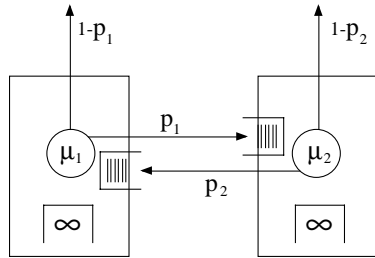


FIGURE 1: A two node Jackson network with an infinite supply of work

In Table 1 we list the values of the stationary probabilities for such a symmetric two node Jackson network with an infinite supply of work, with $\mu_1 = \mu_2$, $p_1 = p_2 = 0.5$, and $\rho_1 = \rho_2 = 0.5$. The occupancies of the two queues are clearly not independent under this distribution. For comparison we give in Table 2 the corresponding probabilities of a standard symmetric two node product form Jackson network with $\rho = 0.5$. The calculation uses the results of [1].

TABLE 1: *Stationary distribution of the infinite supply Jackson network*

$n_2 \quad n_1$	0	1	2	3	4
0	0.151946	0.151946	0.094020	0.050354	0.025752
1	0.151946	0.057926	0.021833	0.009457	0.004471
2	0.094020	0.021833	0.005688	0.001918	0.000802
3	0.050354	0.009457	0.001918	0.000481	0.000161
4	0.025752	0.004471	0.000802	0.000161	0.000040

The following intuitive thought may provide an explanation to the dependence between the two nodes in the system of Figure 1: If we observe a large number of items in node 1, this indicates that there were more arrivals than departures in the recent past. Arrivals come from node 2, and departures go to node 2, so this may indicate more departures and less arrivals at node 2 in the recent past. Thus observing many customers in node 1 may indicate a small number of customers in node 2. This form of negative correlation is indeed observed in Table 1.

It is a challenging question to derive similar results in the general case.

TABLE 2: *Product form joint distribution*

$n_2 \backslash n_1$	0	1	2	3	4
0	0.25	0.125	0.0625	0.03125	0.015625
1	0.125	0.0625	0.03125	0.015625	0.007813
2	0.0625	0.03125	0.015625	0.007813	0.003906
3	0.03125	0.015625	0.007813	0.003906	0.001953
4	0.015625	0.007813	0.003906	0.001953	0.000977

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