SYMMETRIC STRONG DUALITY FOR A CLASS OF CONTINUOUS LINEAR PROGRAMS WITH CONSTANT COEFFICIENTS

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Abstract. We consider Continuous Linear Programs over a continuous finite time horizon $T$, with linear cost coefficient functions and linear right hand side functions and a constant coefficient matrix, where we search for optimal solutions in the space of measures or of functions of bounded variation. These models generalize the Separated Continuous Linear Programming models and their various duals, as formulated in the past by Anderson, by Pullan, and by Weiss. We present simple necessary and sufficient conditions for feasibility. We formulate a symmetric dual and investigate strong duality by considering discrete time approximations. We prove that under a Slater type condition there is no duality gap and there exist optimal solutions which have impulse controls at 0 and $T$ and have piecewise constant densities in $(0,T)$. Moreover, we show that under non-degeneracy assumptions all optimal solutions are of this form, and are uniquely determined over $(0,T)$.

Key words. Continuous linear programming, symmetric dual, strong duality.

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1. Introduction. We consider problems of the form:

$$\max \int_0^T (\gamma + (T-t)c)^T dU(t)$$

M-CLP s.t. $AU(t) \leq \beta + bt$, $0 \leq t \leq T$, $U(t) \geq 0$, $U(t)$ non-decreasing and right continuous on $[0,T]$. (1)

where $A$ is a $K \times J$ constant matrix, $\beta, b, \gamma, c$ are constant vectors of corresponding dimensions, the integrals are Lebesgue-Stieltjes, $U$ are $J$ unknown functions over the time horizon $[0,T]$, and by convention we take $U(0−) = 0$.

We formulate a symmetric dual problem

$$\min \int_{0−}^T (\beta + (T-t)b)^T dP(t)$$

M-CLP∗ s.t. $A^T P(t) \geq \gamma + ct$, $0 \leq t \leq T$, $P(t) \geq 0$, $P(t)$ non-decreasing and right continuous on $[0,T]$. (2)

with $K$ unknown dual functions $P$ with the same convention $P(0−) = 0$. It is convenient to think of dual time as running backwards, so that $P(T−t)$ corresponds to $U(t)$.

Our main results in this paper include the following:

− We discuss how this formulation relates to and generalizes previous continuous linear programs.
− We show weak duality and present a simple necessary and sufficient test for feasibility of M-CLP. We also present a simple Slater type condition based on the same test.
− We show that under Slater type condition there is no duality gap between M-CLP and M-CLP∗, by considering discrete time approximations. We also show that in this case M-CLP and M-CLP∗ posses optimal solutions.

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We further show that in that case there exist optimal solutions for which \( U(t) \) and \( P(t) \) have impulse controls at 0, T and are absolutely continuous inside \((0, T)\), with piecewise constant densities.

Finally, under appropriate simple non-degeneracy assumptions we show that all optimal solutions are of this form, and that the absolutely continuous part on \((0, T)\) is uniquely determined.

Further research to develop a simplex-type algorithm that constructs solutions of this form is in progress.

We note that the question of existence of strong duality, and whether symmetric dual formulations are useful is far from simple when dealing with linear programs in infinite dimensional spaces [7, 20]. Our results in this paper furnish an example where indeed strong duality can hold with a symmetric dual, if a Slater condition is satisfied.

2. Background and motivation. Continuous linear programs were introduced by Bellman in 1953 [8, 9] to model economic processes: find a bounded measurable \( u \) which

\[
\max \int_0^T c^T u(t) \, dt
\]

Bellman-CLP s.t. \( H(t)u(t) + \int_0^t G(s,u(s)) \, ds \leq a(t), \)
\[
\begin{align*}
  u(t) &\geq 0, \quad t \in [0, T]. \\
  u(t) &\geq 0, \quad t \in [0, T].
\end{align*}
\]

Where \( G(s,t), H(t) \) are given matrix functions. These problems were investigated by Dantzig and some of his students, to model continuous time Leontief systems, and by several other early authors [10, 11, 13, 21, 22], with many publications since, but up to date no efficient algorithms or coherent theory have emerged, and these problems are considered very hard.

Separated continuous linear programs (SCLP) were introduced by Anderson [1, 2] in the context of job-shop scheduling:

\[
\max \int_0^T c^T u(t) \, dt
\]

Anderson-SCLP s.t. \( \int_0^t Gu(s) \, ds \leq a(t), \)
\[
\begin{align*}
  Hu(t) &\leq b(t), \\
  u(t) &\geq 0, \quad t \in [0, T].
\end{align*}
\]

where \( G, H \) are constant matrices, and \( a(t), b(t), c(t) \) are given vector functions. Some special cases of SCLP were solved by Anderson and Philpott [4, 5], and this research and related earlier work were summarized in the 1987 book of Anderson and Nash [3], which also contains many references to work on CLP up to that date.

Major progress in the theory of SCLP was achieved by Pullan [6], [14]–[19]. Pullan considered SCLP problems with \( a(t), b(t) \) and \( c(t) \) piecewise analytic, and formulated a non-symmetric dual to (4) (here we modify Pullan’s original version by letting the dual run in reversed time, as in (2)):
Pullan showed that when the feasible region of $Hu(t) \leq b(t)$ is bounded strong duality holds between (4) and (5). In the special case that $a(t), c(t)$ are piecewise linear and $b(t)$ piecewise constant Pullan provided an infinite but convergent algorithm to solve the problems and observed that $P$ was absolutely continuous, except for atoms at the breakpoints of $a, b, c$.

The results of Pullan raised several questions:
- Is the boundedness restriction necessary?
- Can one formulate a symmetric dual?
- Do solutions of the form observed by Pullan always exist?

More recently Weiss \[24\] considered the following SCLP problem

$$\max \int_0^T (\gamma + (T-t) c)^T u(t) + d^T x(t) \, dt$$

SCLP s.t.

\[
\begin{align*}
Gu(s) \, ds + Fx(t) &\leq \alpha + at \\
Hu(t) &\leq b \\
x(t), u(t) &\geq 0, \quad 0 \leq t \leq T.
\end{align*}
\]

and the symmetric dual

$$\min \int_0^T (\alpha + (T-t) a)^T p(t) + b^T q(t) \, dt$$

SCLP* s.t.

\[
\begin{align*}
G^T p(s) \, ds + H^T q(t) &\geq \gamma + ct \\
F^T p(t) &\geq d \\
q(t), p(t) &\geq 0, \quad 0 \leq t \leq T.
\end{align*}
\]

with constant vectors and matrices $G, F, H, \alpha, a, b, \gamma, c, d$. In contrast to previous work Weiss developed a simplex type algorithm which solves this pair of problems exactly, in a finite bounded number of steps, without using discretization.

The simplex type algorithm of Weiss can solve any pair of problems (6), (7) which possess optimal solutions $u(t), p(t)$ that are bounded measurable functions. It produces solutions with $u(t), p(t)$ piecewise constant, and $x(t), q(t)$ continuous piecewise linear. However, there exist problems for which both (6) and (7) are feasible but either (6) or (7) or both do not possess optimal solutions $u(t), p(t)$ in the space of bounded measurable functions. Moreover, one could construct example, where (6) possess optimal solutions in the space of bounded measurable functions, but (7) is infeasible. Such problems cannot be solved by the algorithm of Weiss. This raises the question whether they can be solved in the space of measures, and motivates our formulation of M-CLP, M-CLP* problems (1), (2).

**DEFINITION 2.1.** Consider Weiss’ SCLP problem (6). Then M-CLP problem with following data:

$$A = \begin{bmatrix} G & 0 & F & -F \\ 0 & -I & I \\ H & I & 0 & 0 \\ -H & -I & 0 & 0 \end{bmatrix} \quad U = \begin{bmatrix} U_u(t) \\ U^+_s(t) \\ U^-_u(t) \end{bmatrix} \quad \beta^* + b^* t = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} t,$$

called M-CLP extension of SCLP.

**THEOREM 2.2.** M-CLP/M-CLP* are generalizations of Weiss’ SCLP/SCLP* in the following sense:
(i) if SCLP (6) and SCLP* (7) possess optimal solutions, then these solutions are also optimal solutions of corresponding M-CLP/M-CLP* extensions.

(ii) if Slater type condition 3.3 holds for M-CLP/M-CLP extensions, then for optimal values V following holds:

\[ V(\text{SCLP}) \leq V(M-\text{CLP}) = V(M-\text{CLP}) \leq V(\text{SCLP}^*) \]

(we consider optimal value of infeasible SCLP $-\infty$ and optimal value of infeasible SCLP* $+\infty$)

(iii) if in addition to (ii) SCLP or SCLP* or both are feasible then (ii) still holds if corresponding inequalities are replaced by equalities.

**Proof.** (i) Consider an optimal solution \( x^*(t), u^*(t) \) of (6). By the Structure Theorem (Theorem 3 in [24]) \( x^*(t) \) is absolutely continuous and hence of bounded variation. Therefore we can write \( x^*(t) \) as the difference of two non-decreasing functions \( x^*(t) = U^+(t) - U^-(t) \). Let \( u_i(t) \) be the slacks of the constraints \( Hu(t) \leq b \), and let \( U(t) = \int_0^t u^i(t) dt \), \( U_i(t) = \int_0^t u_i(t) dt \). Then \( \partial = [U, U_1, U^+, U^-] \) uniquely determine \( x^*, u^* \), and satisfy the constraints of the M-CLP extension. One can see that under this reformulation the optimal solution of SCLP is a feasible solution of M-CLP extension with the same objective value.

A similar argument applies to an optimal solution \( q^*(t), p^*(t) \) of (7), which can be then reformulated as the M-CLP* extension which is dual to the M-CLP, with the same objective values.

Weak duality of M-CLP and M-CLP* (see Proposition 3.1 below) then shows that these solutions are the optimal solutions of M-CLP and M-CLP*.

(ii) Consider following CLP problem:

\[
\begin{align*}
\min & \quad \int_0^T (\alpha + (T - t)a)^T dP(t) + \int_0^T b^T q(t) dt \\
\text{DCLP s.t.} & \quad G^T P(t) + H^T q(t) \geq \gamma + ct \\
& \quad F^T P(t) + P^j(t) \geq dt \\
& \quad -F^T P(t) - P^j(t) \geq -dt \\
& \quad q(t) \geq 0, \quad P(t), P^j(t) \text{ non-decreasing and right continuous on } [0,T].
\end{align*}
\]

which is a generalization of Pullans’ dual for the case when \( a(t), c(t) \) are affine, and \( b(t) \) is constant.

One can see that weak duality holds between SCLP (6) and DCLP, and also between M-CLP extension and DCLP. But under Slater type condition for M-CLP extension exists primal and dual optimal solution \( \hat{U}(t), \hat{P}(t) \) with no duality gap (see Theorem 4.7 below). Now, setting \( P(t) = \hat{P}_j(t), P^j(t) = \bar{P}_j(t), q(t) = \hat{P}_j(t) - \bar{P}_j(t) \) we obtain feasible solution of DCLP with same objective value. This solution is optimal by weak duality between M-CLP extension and DCLP. Then, by weak duality between SCLP and DCLP \( V(\text{SCLP}) \leq V(M-\text{CLP}) = V(M-\text{CLP}) \).

Finally, by weak duality between SCLP and SCLP* (Proposition 1 in [24]) we have \( V(\text{SCLP}) \leq V(\text{SCLP}^*) \).

(iii) Let SCLP be feasible, then one can find feasible solution \( u^*(t), x^*(t) \), where \( x^*(t) \) is of bounded variation. Consider now additional constraint \( u(t) \leq W, 0 \leq t \leq T, W \geq \max_{j,0 \leq s < T} u^j(t) \), which preserving SCLP to be feasible. Moreover, this solution could be translated to solution of M-CLP extension as shown in proof of (i). Hence, M-CLP extension also still be feasible if we add equivalent constraint to it. Moreover, one could choose \( W \) big enough to preserve Slater type condition.
Now, consider \( \tilde{U}(t) \) be optimal solution of M-CLP extension with additional constraint. It is clear, that for this solution \( \tilde{u}^0 = \tilde{u}^N = \tilde{u}^0_N = \tilde{u}^N_s = 0 \), and hence this solution could be translated back to the optimal solution of SCLP by following:

\[
\begin{align*}
  u^{**} &= \frac{d\tilde{U}}{dt} \\
x^{**} &= U^+(t) - U^-(t)
\end{align*}
\]

Finally, consider sequence \( W^{(n)}(n) = nW, n = 1, \ldots \) and let \( D^{(n)}(n) \) be sequence of optimal solutions of M-CLP extension with additional constraint, and \( u^{(n)}, x^{(n)} \) be sequence of corresponding optimal solution of SCLP with additional constraint. It is clear that feasible region growth in \( n \), and hence sequences of objective values \( V(SCLP)^{(n)} = V(M - CLP)^{(n)} \) involving by corresponding solutions are increasing. Hence letting \( n \to \infty \) we obtain:

\[
\lim_{n \to \infty} V(SCLP)^{(n)} = \lim_{n \to \infty} V(M - CLP)^{(n)} = V(M - CLP)
\]

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\[
\lim_{n \to \infty} V(SCLP)^{(n)} = \lim_{n \to \infty} V(M - CLP)^{(n)} = V(M - CLP)
\]

It is not hard to see that (1), (2) generalize also Anderson and Pullan’s problems (4), (5) restricted to \( a(t), c(t) \) affine, and \( b(t) \) constant.

3. Weak duality, complementary slackness and feasibility.

**Proposition 3.1.** Weak duality holds for M-CLP, M-CLP* (1),(2).

**Proof.** Let \( U(t), P(t) \) be feasible solutions for (1), (2), and compare their objective values:

**Dual objective**

\[
\begin{align*}
  &= \int_0^T (\beta + (T-t)b)^T dP(t) \\
  &\geq \int_0^T \left( \int_0^{T-t} AdU(s) \right)^T dP(t) \\
  &= \int_0^T \left( \int_0^{T-t} A^T dP(s) \right)^T dU(t) \\
  &\geq \int_0^T (\gamma + (T-t)c)^T dU(t)
\end{align*}
\]

= Primal objective.

The first inequality follows from the primal constraints at \( T - t \), and from \( P(t) \) non-decreasing. The equality follows by changing order of integration, using Fubini’s theorem. The second inequality follows from the dual constraints at \( T - t \), and from \( U(t) \) non-decreasing.

Equality of the primal (M-CLP) and dual (M-CLP*) objective will occur if and only if the following holds:

**Complementary slackness condition.** Let \( x(t) = \beta + bt - AU(t) \) and \( q(t) = A^T P(t) - \gamma - ct \) be the slacks in (1), (2). The complementary slackness condition for M-CLP, M-CLP* is

\[
\int_0^T x(T-t)^T dP(t) = \int_0^T q(T-t)^T dU(t) = 0.
\]

In the following propositions and theorems in this and following sections we present results for M-CLP. By symmetry these results hold for M-CLP*, with the obvious modifications.
We present now a simple necessary and sufficient condition for feasibility. This is similar to a condition derived by Wang, Zhang and Yao [23]. It involves the standard linear program Test-LP and its dual Test-LP*.

\[
\max z = (\gamma + cT)^T u + \gamma^T U
\]

Test-LP s.t. \( Au \leq \beta \)

\( Au + AU \leq \beta + bT \)

\( u, U \geq 0 \) \hspace{1cm} (10)

**Proposition 3.2.** M-CLP is feasible if and only if Test-LP is feasible.

**Proof.** (i) **Sufficiency:** Let \( u, U \) be a solution of Test-LP (10), with slacks \( x^0 = \beta - Au, x^T = \beta + bT - Au - AU \). Then \( U(t) = u + \frac{t}{T} U, 0 \leq t \leq T \) is a feasible solution of M-CLP (1), with non-negative slacks \( x(t) = (1 - \frac{t}{T}) x^0 + \frac{t}{T} x^T \). To check this we have for \( 0 \leq t \leq T \):

\[
AU(t) + x(t) = A \left( u + \frac{t}{T} U \right) + \left( 1 - \frac{t}{T} \right) x^0 + \frac{t}{T} x^T
\]

\[
= \frac{t}{T} (Au + AU + x^T) + \left( 1 - \frac{t}{T} \right) (Au + x^0)
\]

\[
= \beta + bt.
\]

(ii) **Necessity:** Let \( U(t) \) be a feasible solution of M-CLP (1) with slack \( x(t) \geq 0 \). Then \( u = U(0), U = \int_0^T dU(t) \) with slack \( x^0 = x(0), x^T = x(T) \) is a feasible solution for Test-LP (10), as is seen immediately. \( \Box \)

We use the following definition:

**Definition 3.3 (Slater type condition).** We say that the Test-LP problem (10) is strictly feasible at \( T \) if there exists a feasible solution \( u, U \) of (10) and a constant \( \alpha > 0 \) such that \( \beta - Au \geq \alpha \) and \( \beta + bT - Au - AU \geq \alpha \). We say that M-CLP is strictly feasible at \( T \) if there exists a feasible solution \( U(t) \) of (1) and a constant \( \alpha > 0 \) such that \( \beta + bt - AU(t) \geq \alpha \) for all \( t \in [0, T] \).

**Corollary 3.4.** M-CLP is strictly feasible if and only if Test-LP is strictly feasible.

**Proof.** Simply define \( \beta^* = \beta - \alpha \) and recall Proposition 3.2 for problems with \( \beta \) replaced by \( \beta^* \). \( \Box \)

4. *Discrete time approximations and strong duality.* In this section we consider a pair of M-CLP/M-CLP* problems which are feasible, and use time discretization to solve them approximately. We prove that if M-CLP and M-CLP* are strictly feasible, then there is no duality gap and an optimal solution exists. We use a discretization approach similar to [14].

4.1. *General discretizations.* For a partition \( 0 = t_0 < t_1 < \ldots < t_N = T \) we define the discretization of M-CLP to be:

\[
\max z = (\gamma + cT)^T u^0 + \sum_{i=1}^{N} \left( \gamma + \left( T - \frac{t_i + t_{i-1}}{2} \right) c \right)^T (t_i - t_{i-1}) u^i + \gamma^T u^N
\]

s.t. \( Au^0 + x^0 = \beta \)

\( dCLP_1 \)

\[
Au^0 + A \sum_{i=1}^{n} (t_i - t_{i-1}) u^i + x^i = \beta + bt_n \quad \text{for} \ n = 1, \ldots, N
\]

\[
Au^0 + A \sum_{i=1}^{N} (t_i - t_{i-1}) u^i + Au^N + x^N = \beta + bT
\]

\( u^0, u^1, \ldots, u^N, u^N, x^0, x^1, \ldots, x^N, x^N \geq 0. \)
and for the same time partition the discretization of M-CLP* is defined as:

\[
\min z = (\beta^T + T^b) p^N + \sum_{i=1}^{N} \left( \beta^T + \frac{t_i + t_{i-1}}{2} b^T \right) (t_i - t_{i-1}) p^i + \beta^T p^0
\]

\[
s.t. A^T p^N - q^N = \gamma
\]

(12)

dCLP2

\[A^T p^N + A^T \sum_{i=n}^{N} (t_i - t_{i-1}) p^i - q^{n-1} = \gamma + c(T - t_{n-1}) \quad \text{for } n = N, \ldots, 1\]

Note that these two problems are not dual to each other.

Following Pullan [14], for a partition 0 = t_0 < t_1 < \ldots < t_N = T and values \(f(t_0), \ldots, f(t_N)\) we define the piecewise linear extension:

\[f_L(t) = \frac{(t_i - t)}{t_i - t_{i-1}} f(t_{i-1}) + \frac{(t - t_{i-1})}{t_i - t_{i-1}} f(t_i) \quad \text{for } t \in [t_{i-1}, t_i) \text{ for } i = 1, \ldots, N.\]

and the piecewise constant extension:

\[f_C(t) = f(t_i), \quad t \in [t_{i-1}, t_i), i = 1, \ldots, N.\]

The following proposition is an easy extension of Proposition 3.2.

PROPOSITION 4.1. All discretizations dCLP1 (11) are feasible if and only if M-CLP is feasible.

Proof. (i) Let \(U(t)\) be a feasible solution of M-CLP (1) with slacks \(x(t) \geq 0\). Then \(u^0 = U(0), u^n = \frac{1}{t_n - t_{n-1}} \int_{t_{n-1}}^{t_n} dU(t)\) (in these integrals we take \(t_0 = 0\) and \(t_n = t_N - \) for \(n = 1, \ldots, N\)), \(u^N = U(T) - U(T^-)\), and \(x^0 = x(0), x^n = x(t_n^-), n = 1, \ldots, N, x^N = x(T)\) is a feasible solution for dCLP1 (11). To check this we have for \(n = 0, \ldots, N:\)

\[A u^0 + A \sum_{i=1}^{n} (t_i - t_{i-1}) u^i + x^n =\]

\[AU(0) + A \sum_{i=1}^{n} (t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} \frac{1}{t_n - t_{n-1}} dU(t) + x(t_n^-) = AU(t_n^-) + x(t_n^-) = \beta + bt_n\]

(ii) Let \(u^0, u^1, \ldots, u^N, u^N\) be a feasible solution of dCLP1. Define \(U(0) = u^0\), let \(u(t)\) be the piecewise constant extension of \(u_1, \ldots, u_N\), and let \(U(t) = U(0) + \int_0^t u(s) ds, t \in [0, T), U(T) = U(T^-) + u^N\). Then \(U(t)\) is a feasible solution of M-CLP.

PROPOSITION 4.2. Any feasible solution of dCLP1 can be extended to a feasible solution of M-CLP with equal objective value.

Proof. We set \(u(t)\) to be the piecewise constant extension of \(u^1, \ldots, u^N\) and take \(U(t)\) to be the measure with density \(u(t)\) on \((0, T)\) and impulses \(U([0]) = u^0, U([T]) = u^N\). We also set \(x(t)\) to be the piecewise linear extension of \(x^0, \ldots, x^N\), and take \(x^N\) to be the same for both problems. It is immediate to see that this gives a feasible solution to M-CLP. Furthermore, it is immediate to see that the objective of dCLP1 equals the objective of M-CLP for this extended solution.

PROPOSITION 4.3. The optimal values \(V\) of the various problems satisfy:

\[V(dCLP_1) \leq V(M-CLP) \leq V(M-CLP^*) \leq V(dCLP_2)\]
Proof. The first and last inequalities follow from Proposition 4.2 and the middle inequality follows from weak duality. □

4.2. Discretizations with equidistant partitions. Similar to Wang, Zhang and Yao [23] and to Pullan [14] we use even equidistant partitions, denoted $\pi^N$ which divides the interval $[0,T]$ into $N$ equal segments, each of length $2\varepsilon$, i.e. $\varepsilon = \frac{T}{2N}$. With this partition we introduce the notations:

- Given a $K \times J$ matrix $A$ we define the $NK \times J$ matrix $A_\parallel$, the $NK \times NJ$ matrix $A_\perp$, and the $K \times NJ$ matrix $A_\equiv$ as follows:

$$A_\parallel = \begin{bmatrix} A & A & \cdots & A \\ A & A & \cdots & A \\ \vdots & \vdots & \ddots & \vdots \\ A & A & \cdots & A \end{bmatrix}, \quad A_\perp = \begin{bmatrix} A \\ A \\ \vdots \\ A \end{bmatrix}, \quad A_\equiv = [A \ A \ \cdots \ A].$$

- We define the $N$-fold vectors, each with $N$ vector components:

$$\hat{\beta} = \begin{bmatrix} \beta \\ \vdots \\ \beta \end{bmatrix}, \quad \hat{\gamma} = \begin{bmatrix} \gamma \\ \vdots \\ \gamma \end{bmatrix}.$$

$$\hat{b}_1 = \begin{bmatrix} 2b\varepsilon \\ 4b\varepsilon \\ \vdots \\ bT \end{bmatrix}, \quad \hat{b}_2 = \begin{bmatrix} b\varepsilon \\ 3b\varepsilon \\ \vdots \\ b(T-\varepsilon) \end{bmatrix}, \quad \hat{c}_1 = \begin{bmatrix} c(T-\varepsilon) \\ c(T-3\varepsilon) \\ \vdots \\ c\varepsilon \end{bmatrix}, \quad \hat{c}_2 = \begin{bmatrix} cT \\ c(T-2\varepsilon) \\ \vdots \\ 2c\varepsilon \end{bmatrix}.$$

$$\Delta U = \begin{bmatrix} \Delta U^1 \\ \vdots \\ \Delta U^N \end{bmatrix}, \Delta P = \begin{bmatrix} \Delta P^1 \\ \vdots \\ \Delta P^N \end{bmatrix}, \hat{x} = \begin{bmatrix} x^1 \\ \vdots \\ x^N \end{bmatrix}, \hat{q} = \begin{bmatrix} q^0 \\ \vdots \\ q^{N-1} \end{bmatrix}.$$

Using this notation we rewrite problems (11), (12) for even equidistant partitions, as:

$$\begin{align*}
\text{max } z &= (\gamma + cT)^T u^0 + (\hat{\gamma} + \hat{c}_1)^T \Delta U + \gamma^T u^N \\
\text{s.t. } Au^0 + x^0 &= \beta \\
\text{dCLP}_1(\pi^N) : \quad A_\parallel u^0 + A_\perp \Delta U + \hat{x} &= \hat{\beta} + \hat{b}_1 \\
A u^0 + A_\equiv \Delta U + A u^N + x^N &= \beta + bT \\
u^0, \Delta U, u^N, x^0, \hat{x}, x^N &\geq 0.
\end{align*}$$

$$\begin{align*}
\text{min } z &= (\beta + bT)^T p^N + (\hat{\beta} + \hat{b}_2)^T \Delta P + \beta^T p^0 \\
\text{s.t. } A^T p^N - q^N &= \gamma \\
\text{dCLP}_2(\pi^N) : \quad A^\parallel p^N + A^\perp \Delta P - \hat{q} &= \hat{\gamma} + \hat{c}_2 \\
A^T p^N + A^\equiv \Delta P + A^T p^0 - q^0 &= \gamma + cT \\
p^N, \Delta P, p^0, q^N, \hat{q}, q^0 &\geq 0.
\end{align*}$$

The reader may notice that in (14) we have for convenience reversed the order of variables and the order of the constraints in the middle part of the problem relative to (12).
To quantify the discretization error for time partition \( \pi^N \) we define a modified pair of problems \( \text{mdCLP}(\pi^N), \text{mdCLP}^*(\pi^*) \):

\[
\text{mdCLP}(\pi^N) \max z = (\gamma + cT)^T u^0 + (\hat{\gamma} + \hat{c}_2)^T \Delta U + \gamma^T u^N \\
\text{s.t. Constraints of (13)}
\]

\[
\text{mdCLP}^*(\pi^N) \min z = (\beta + bT)^T p^N + (\hat{\beta} + \hat{b}_1)^T \Delta P + \beta^T p^0 \\
\text{s.t. Constraints of (14)}
\]

We note that they are dual to each other. Moreover, since (13), (14) are (11), (12) rewritten, the problems \( \text{mdCLP}(\pi^N) \) and \( \text{mdCLP}^*(\pi^N) \) are feasible if and only if M-CLP, M-CLP* are feasible by Proposition 4.1. In this case they also possess optimal solutions. Denote by \( \pi^* \), \( \Delta \) to quantify the discretization error for time partition \( \pi^N \).

**Theorem 4.4.** If M-CLP and M-CLP* are feasible then by solving \( \text{mdCLP}(\pi^N) \) and \( \text{mdCLP}^*(\pi^N) \) the following bounds holds:

\[
V(\text{M-CLP}^*) - V(\text{M-CLP}) \leq V(\text{dCLP}_2(\pi^N)) - V(\text{dCLP}_1(\pi^N)) \leq (N)\epsilon
\]

where

\[
\gamma(N) = c^T \sum_{i=1}^{N} \Delta U^{*i} - b^T \sum_{i=1}^{N} \Delta P^{*i} > 0
\]

**Proof.** The first inequality follows from Proposition 4.3. To evaluate the second inequality we note that the optimal solutions of \( \text{mdCLP}(\pi^N) \) and \( \text{mdCLP}^*(\pi^N) \) are feasible but suboptimal solutions of \( \text{dCLP}_1(\pi^N) \) and \( \text{dCLP}_2(\pi^N) \). Calculating the objective values of \( \text{dCLP}_1(\pi^N) \), \( \text{dCLP}_2(\pi^N) \) for the solutions \( u^0, \Delta U^*, u^N, p^0, \Delta P^*, p^N \) we have:

\[
V(\text{dCLP}_1(\pi^N)) = (\gamma + cT)^T u^0 + (\hat{\gamma} + \hat{c}_1)^T \Delta U^* + \gamma^T u^N,
\]

\[
V(\text{dCLP}_2(\pi^N)) = (\beta + bT)^T p^N + (\hat{\beta} + \hat{b}_2)^T \Delta P^* + \beta^T p^0
\]

(15)

On the other hand, because \( \text{mdCLP}(\pi^N) \) and \( \text{mdCLP}^*(\pi^N) \) are dual problems:

\[
V(\text{mdCLP}(\pi^N)) = (\gamma + cT)^T u^0 + (\hat{\gamma} + \hat{c}_2)^T \Delta U^* + \gamma^T u^N =
\]

\[
= (\beta + bT)^T p^N + (\hat{\beta} + \hat{b}_1)^T \Delta P^* + \beta^T p^0 = V(\text{mdCLP}^*(\pi^N))
\]

(16)

Combining (15) and (16), after easy manipulations we get:

\[
V(\text{dCLP}_2(\pi^N)) - V(\text{dCLP}_1(\pi^N)) \leq (\hat{b}_2 - \hat{b}_1) \Delta P^* + (\hat{c}_2 - \hat{c}_1) \Delta U^* = \epsilon \gamma(N) \quad \square
\]

**Theorem 4.5.** The sequence of optimal values of the dual problems \( \text{mdCLP}(\pi^N) \) and \( \text{mdCLP}^*(\pi^N) \) has finite lower and upper bounds, \( V_L, V_U \).

**Proof.** We consider the single interval partition \( \pi^1 \), where we have the problem:

\[
\max z = (\gamma + cT)^T u + (\gamma + cT)^T U + \gamma^T u^T \\
\text{s.t.} \quad Au + x^0 = \beta
\]

\[
\text{mdCLP}(\pi^1) \quad Au + AU + x^T = \beta + bT \quad (17)
\]

\[
Au + AU + Au^T + x^T = \beta + bT \\
u, u^T, U, x^0, x^T \geq 0.
\]
An optimal solution to (17) can be extended to a feasible solution of mdCLP($\pi^N$) as follows:

$$u^0 = u, \ u^N = u^T, \ \Delta U = \left[ \frac{U}{N}, \ldots, \frac{U}{N} \right], \ \hat{x} = \left[ (1-2\varepsilon)x^0 + 2\varepsilon x', \ldots, (1-2\varepsilon)x' + 2\varepsilon x^0 \right],$$

$$x^0 = x^0, \ x^N = x^T.$$ Hence the following inequality holds:

$$V(\text{mdCLP}(\pi^N)) \geq (\gamma + cT)u + \gamma^TU + \gamma^T\Delta U + \gamma^Tu^T$$

$$= (\gamma + cT)u + \gamma^TU + c^T \left( \frac{T}{2} + \varepsilon \right) U + \gamma^Tu^T$$

$$\geq (\gamma + cT)u + \gamma^TU + c^T \left( \frac{T}{2} - c^{-T} \right) U + \gamma^Tu^T = V_L$$

where $c^* = \max(c_j, 0)$, $c^- = \max(-c_j, 0)$, and we recall that $\varepsilon \leq \frac{T}{2}$.

Similarly, by considering the dual, an upper bound is obtained in terms of the solution of the dual test problem:

$$V_U = (\beta + bT)^T p + \beta^TP + \left( b^T - b \frac{T}{2} \right)^TP + \beta^TP^T$$

4.3. Bounding the discrete solutions. In this section we assume that M-CLP as well as M-CLP* satisfy the Slater type condition 3.3. Under this assumption we will show that all the optimal solutions of mdCLP(\(\pi^N\)) and mdCLP*(\(\pi^N\)) are uniformly bounded.

We consider first the sequence of primal problems \(\{\text{mdCLP}(\pi^N)\}_{N=1}^\infty\). We use the following notations:

\[
\{u^{0*(N)}, U^{*(N)}, u^{N*(N)}\}_{N=1}^\infty, \text{ are the optimal solutions,}
\]

\[
u^{*(N)}(t_i) = \Delta U^{*(N)/2\varepsilon}, \quad i = 1, \ldots, N,
\]

\[
u^{*(N)}(t) \text{ is the piecewise constant extension of the } u^{*(N)}(t_1), \ldots, u^{*(N)}(t_N)
\]

\[
U^{*(N)}(t) = u^{0*(N)} + \int_0^t u^{*(N)}(s)ds, \quad t \in [0, T), \quad U^{*(N)}(T) = U^{*(N)}(T-) + u^{N*(N)}.
\]

**Theorem 4.6.** If M-CLP* is strictly feasible then all $J$ elements of $U^{*(N)}(T)$ have a uniform finite upper bound.

**Proof.** Take any $j = 1, \ldots, J$, we will show that $U_j^{*(N)}(T)$ is bounded by a constant $\Psi_j$ for all $N$. Recall that $U^{*(N)}(t)$ are non decreasing, so this bound will hold for all $U_j^{*(N)}(t), t \in [0, T]$.

We choose $N_0$ large enough and corresponding $\varepsilon$ small enough so that:

$$\varepsilon \leq \frac{\alpha_1}{2} \implies \frac{T}{2N_0} \leq \frac{T}{2N_0} \implies N_0 \geq \frac{T}{\alpha_1}$$

where $\alpha_1$ is a small constant, to be determined later. We will find a uniform bound for $U_j^{*(N)}(T), N \geq N_0$.

We use the following notation:

$$\delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_J \end{bmatrix} \text{ where } \delta_j = \begin{cases} \frac{\alpha_1}{2} & \text{if } c_j > 0 \\ 0 & \text{if } c_j \leq 0 \end{cases},$$

$$\hat{c} = \begin{bmatrix} c \\ \vdots \end{bmatrix} \text{ and } \delta = \begin{bmatrix} \delta \\ \vdots \end{bmatrix} \text{ are the } N\text{-fold vectors of } c\text{’s and } \delta\text{’s}.$$
Consider the following discrete linear optimization problem, for a discrete error bound:

\[
\Psi^{(1)}_j(N) = \max \frac{\alpha_1}{2} \mathbf{u}_j^0 + \frac{\alpha_1}{2} \sum_{i=1}^N \Delta U_j^i + \frac{\alpha_1}{2} \mathbf{u}_j^N
\]

dEBLP(\pi^N) \quad \text{s.t.} \quad (\gamma + \delta + cT)\mathbf{u}^0 + (\hat{\gamma} + \hat{\delta} + \hat{c})^T \Delta U + (\gamma + \delta)\mathbf{u}^N \geq V_L \quad (18)

Constraints of mdCLP(\pi^N)

One can see that \(\hat{c}_1 + \hat{\delta} \geq \hat{c}_1 + \hat{c}e = \hat{c}_2\) and hence, by Theorem 4.5 the first constraint of dEBLP(\pi^N) holds for the solution \(\{\mathbf{u}^{0+}(N), \Delta U^{+}(N), \mathbf{u}^{N+}(N)\}\). Hence the optimal solution of mdCLP(\pi^N) is feasible for dEBLP(\pi^N). In particular, it follows that \(\Psi^{(1)}_j(N) \geq U^{+}_{j}(T)\)

The problem dEBLP(\pi^N) is a discretization of the following continuous linear programming problem:

\[
\Psi^{(2)}_j = \max \frac{\alpha_1}{2} \int_0^T dU_j(t) dt
\]

EBCLP \quad \text{s.t.} \quad \int_0^T (\gamma + \delta + (T-t)c)^T dU(t) \geq V_L \quad (19)

Constraints of M-CLP

The continuous linear program EBCLP is not formulated exactly as an M-CLP problem, the difference being that the first constraint has linearly varying coefficients rather than constant coefficients. Nevertheless, one can show by similar arguments that propositions 4.1, 4.2 still hold, and so for every \(N\), \(U^{+}(N)(t)\) is a feasible solution of EBCLP. We now have that \(\Psi^{(2)}_j \geq \Psi^{(1)}_j\).

EBCLP is obviously feasible. We now need to show that it is bounded. We formulate the following dual problem to EBCLP:

\[
\Psi^{(3)}_j = \min \int_0^T (\beta + (T-t)b)^T dP(t) - V_L P^O
\]

EBCLP* \quad \text{s.t.} \quad A^T P(t) \geq (\gamma + \delta + ct)P^O + \frac{\alpha_1}{2} \mathbf{e}^j, \quad 0 \leq t \leq T, \quad P^O \geq 0, P(t) \geq 0, \text{ non-decreasing and right continuous on } [0,T].

where \(\mathbf{e}^j\) is the \(j\)th unit vector.

It is straightforward to check that weak duality holds between problems EBCLP and EBCLP*. Hence, if EBCLP* is feasible, we have \(\Psi^{(3)}_j \geq \Psi^{(2)}_j\).

It remains to show that EBCLP* is feasible. We now use the assumption that M-CLP* is strictly feasible. Hence there exists a vector of functions \(\check{P}(t), t \in [0,T]\) that satisfy:

\[
A^T \check{P}(t) \geq \gamma + ct
\]

\[
A^T \check{P}(t) \geq \gamma + ct + \alpha_1,
\]

\[
\check{P}(t) \geq 0, \text{ non-decreasing and right continuous on } [0,T].
\]

for some small enough value \(\alpha_1\). This gives us our choice for the value of \(\alpha_1\).

It is now easy to check that \(P^O = 1\) and \(\check{P}(t), t \in [0,T]\) is a feasible solution of EBCLP*, indeed, for \(P^O = 1\):

\[
A^T \check{P}(t) \geq \gamma + ct + \alpha_1 \geq (\gamma + \delta + ct)P^O + \frac{\alpha_1}{2} \mathbf{e}^j, \quad 0 \leq t \leq T.
\]
Let $\Psi_j^{(4)}$ be the value of the objective of EBCLP* for this solution. We have: $\Psi_j^{(4)} \geq \Psi_j^{(3)} \geq \Psi_j^{(2)} \geq \Psi_j^{(1)} \geq U_j^{*(N)}(T), N \geq N_0$.

Finally:

$$\Psi_j = \max \{ \Psi_j^{(4)}, U_j^{*(N)}(T), N = 1, \ldots, N_0 \} \geq U_j^{*(N)}(T) \text{ for all } N.$$  

Let $P^{*(N)}(t)$ be defined for the optimal solutions of mdCLP ($\pi^N$), similar to $U_j^{*(N)}$. A similar proof shows that if M-CLP is strictly feasible, we can construct for any $k = 1, \ldots, K$ a bound: $\Phi_k \geq \Phi_k^{(N)}(T)$.

4.4. Strong duality.

**Theorem 4.4.** If M-CLP and M-CLP* are strictly feasible, then both have optimal solutions, and there is no duality gap.

**Proof.** We show first that there is no duality gap. In Theorem 4.4 we have seen that

$$V(M-CLP^*) - V(M-CLP) \leq V(dCLP_2(\pi^N)) - V(dCLP_1(\pi^N)) \leq \Upsilon(N) \varepsilon$$

where

$$\Upsilon(N) = c^T U^{*(N)}(T) - b^T P^{*(N)}(T)$$

In Theorem 4.6 we saw that all components of $U^{*(N)}(T), P^{*(N)}(T)$ are uniformly bounded, by quantities $\Psi_j, \Phi_k$. We therefore have a uniform bound $\Upsilon$:

$$\Upsilon(N) \leq \Upsilon = \sum_{j=1}^{J} c_j^+ \Psi_j + \sum_{k=1}^{K} b_k^- \Phi_k,$$

where $c_j^+ = \max(c_j, 0), b_k^- = \max(-b_k, 0)$. Hence,

$$0 \leq V(M-CLP^*) - V(M-CLP) \leq \varepsilon \Upsilon$$

and letting $N \to \infty$, so that $\varepsilon \to 0$, we get $V(M-CLP^*) = V(M-CLP)$.

We next show that optimal solutions exist. We saw that $U^{*(N)}(t)$ are feasible solutions for M-CLP for all $N$. $U^{*(N)}(t)$ are vectors of non-negative non-decreasing functions, and by Theorem 4.6 they are all uniformly bounded. By Helly’s selection principle (Theorem 5, p. 372 in [12]), it is then possible to find a subsequence $N_m$ such that $U_j^{*(N_m)}(t)$ converge pointwise for every $t$ to a non-negative non-decreasing right continuous function $U_j(t), t \in [0, T]$, for all $j = 1, \ldots, J$. It is immediate to see that $U(t)$ is a feasible solution for M-CLP.

By Helly’s convergence theorem (Theorem 4, p. 370 in [12]) by the continuity of $c^T(T-t)$

$$\lim_{N_m \to \infty} \int_0^T (\gamma + (T-t)c)^T dU^{(N_m)}(t) = \int_0^T (\gamma + (T-t)c)^T dU(t), \quad (22)$$

but this limit equals $V(M-CLP)$, hence $U(t)$ is an optimal solution of M-CLP. Similarly the dual problem M-CLP* has an optimal solution.  

5. Form of optimal solution. We now consider that M-CLP problem has an optimal primal solution $U_0(t)$. In particular, this is true if M-CLP are primal and dual strictly feasible (see Theorem 4.7). In this section we investigate properties of the optimal solution.

**Proposition 5.1.** $c^T U_0(t)$ is continuous on $(0, T)$. 

exists an interval \((\tilde{t}, t)\) with \(c^TU_O(t) = c^TU_O(t^-) - c^TU_O(t^+) \neq 0\).

Assume first that \(c^TU_O^T > 0\). Let \(t_a < t_c\) be a point such that \(|c^TU_O(t^-) - c^TU_O(t)| < \frac{1}{4}c^TU_O^T\) for all \(t \in [t_a, t_c]\). Such a point exists because \(U_O\) has left and right limits at \(t_c\). Consider the following solution of M-CLP:

\[
\tilde{U}(t) = \begin{cases} 
U_O(t_a) + \frac{t-t_a}{t_c-t_a}(U_O(t_c+) - U_O(t_a)), & t \in [t_a, t_c], \\
U_O(t), & t \notin [t_a, t_c].
\end{cases}
\]

It is clear that \(\tilde{U}(t)\) is feasible. Comparing the objective values for \(U_O(t)\) and \(\tilde{U}(t)\) we obtain:

\[
\begin{align*}
\int_0^T (\gamma^T + c^T(T-t))d\tilde{U}(t) &- \int_0^T (\gamma^T + c^T(T-t))dU_O(t) \\
= &\int_{t_a}^{t_c} \int_{t_a}^{t_c} c^t dU_O(t) - \int_{t_c}^{t_c} c^t d\tilde{U}(t) \\
= &\frac{t_c - t_a}{2} \left( c^TU_O^T + c^TU_O(t^-) + c^TU_O(t_a) \right) - \int_{t_a}^{t_c} c^T U_O(t)dt \\
\geq &\frac{t_c - t_a}{2} \left( c^TU_O^T - \sup_{t \in [t_a, t_c]} t^TU_O(t) - c^TU_O(t_a) - c^TU_O(t^-) \right) > 0
\end{align*}
\]

where in the second equality we replace Lebesgue-Stieltjes integral by Riemann-Stieltjes integral and integrate by parts. This contradicts the optimality of \(U_O(t)\). A similar contradiction is obtained if \(c^TU_O^T < 0\), considering a point \(t_a > t_c\). We conclude that \(c^TU_O(t)\) has no jumps, and hence is continuous on \((0, T)\). \(\square\)

**Proposition 5.2.** \(c^TU_O(t)\) is concave on \((0, T)\).

**Proof.** Assume the contrary. Then, since by Proposition 5.1 \(c^TU_O(t)\) is continuous, there exists an interval \((t_1, t_2)\) such that:

\[
c^TU_O(t) < c^TU_O(t_1) + \frac{t-t_1}{t_2-t_1}(c^TU_O(t_2) - c^TU_O(t_1)), \quad t \in (t_1, t_2)
\]

Consider the following solution of M-CLP:

\[
U^*(t) = \begin{cases} 
U_O(t_1) + \frac{t-t_1}{t_2-t_1}(U_O(t_2) - U_O(t_1)), & t \in (t_1, t_2), \\
U_O(t), & t \notin (t_1, t_2).
\end{cases}
\]

It is clear that \(U^*(t)\) is feasible. We note also that from our assumption it follows that \(c^TU_O(t) < c^TU^*(t)\) on \((t_1, t_2)\). Comparing objective values for \(U_O(t)\) and \(U^*(t)\) we obtain, similar to the proof of Proposition 5.1:

\[
\begin{align*}
\int_0^T (\gamma^T + c^T(T-t))dU^*(t) &- \int_0^T (\gamma^T + c^T(T-t))dU_O(t) \\
= &\int_{t_1}^{t_2} c^t dU_O(t) - \int_{t_1}^{t_2} c^t dU^*(t) \\
= &\int_{t_1}^{t_2} c^TU^*(t)dt - \int_{t_1}^{t_2} c^TU_O(t)dt > 0.
\end{align*}
\]

This contradicts the optimality of \(U_O(t)\). Hence \(c^TU_O(t)\) is concave. \(\square\)
By the Lebesgue decomposition theorem any feasible solution of M-CLP can be represented as \( U(t) = U_a(t) + U_s(t) \), where \( U_a(t) \) is an absolutely continuous function and \( U_s(t) \) is a singular function, including a discrete singular ("jump") part and a continuous singular part.

**Proposition 5.3.** Consider an optimal solution \( U_0(t) \) and its Lebesgue decomposition \( U_0(t) = U_a(t) + U_s(t) \), and let \( u(t) = \frac{dU_0(t)}{dt} \). Then the following holds:

**(i)**
\[
\frac{d}{dt} c^T U_0(t) = c^T u(t)
\]

**(ii)** \( c^T u(t) \) is a non-increasing function.

**(iii)**
\[
\int_0^T (\gamma + (T-t)c)^T dU_0(t) = \gamma^T U_0(T) + c^T TU_0(T-) - \int_0^T c^T u(t) dt
\]

**Proof.** (i) By Proposition 5.2 \( c^T U_0(t) \) is concave and hence is absolutely continuous on the interval \((0, T)\). Therefore, by uniqueness of the Lebesgue decomposition, \( c^T(U_0(t) - U_0(0)) = c^T U_a(t) \) on this interval.

(ii) That \( c^T u(t) \) is non-increasing follows from the concavity of \( c^T U_0(t) \).

(iii) Because \( (\gamma + (T-t)c)^T \) is continuous the Lebesgue-Stieltjes integral above can be replaced by the Riemann-Stieltjes integral.

\[
\int_0^T (\gamma + (T-t)c)^T dU_0(t) = \gamma^T U_0(T) + c^T TU_0(T-) - \int_0^T c^T u(t) dt
\]

For part (iii) of the next theorem we need the following non-degeneracy assumption:

**Assumption 5.4.** The vector \( c \) is in general position to the matrix \([A' I]\) (it is not a linear combination of any less than \( J \) columns).

**Theorem 5.5.** Assume that M-CLP/M-CLP* have optimal solutions \( U_O(t), P_O(t) \) with no duality gap, then:

(i) \( c^T U_O(t) \) is piecewise linear on \((0, T)\) with a finite number of breakpoints.

(ii) There exists an optimal solution \( U^*(t) \) that is continuous and piecewise linear on \((0, T)\).

(iii) Under the non-degeneracy assumption 5.4, every optimal solution is of this form, and furthermore, \( U_O(t) - U_O(0+) \) is unique over \((0, T)\).

**Proof.** (i) By the Lebesgue differentiation theorem \( U_O(t), P_O(t) \) can be differentiated at least almost everywhere. Let \( S \) be the set where \( U_O(t) \) is not differentiable and \( S' \) be the set where \( P_O(T-t) \) is not differentiable. Let \( E_0 = [0, T] \setminus (S \cup S') \). Then the complementary slackness condition (9) can be rewritten as:

\[
\int_{E_0} q(T-t)^T u(t) dt = \int_{S \cup S'} q(T-t)^T dU_0(t) = \int_{E_0} x(T-t)^T p(t) dt = \int_{S \cup S'} q(T-t)^T dP_0(t) = 0
\]
where $u(t) = \frac{dU(t)}{dt}$, $p(t) = \frac{dp(t)}{dt}$ are the densities of $U(t), P(t)$ on $E$ and $\dot{x}(t) = \beta + bt - A U(t), q(t) = A^T P(t) - \gamma - ct$ are slack functions. Hence, we must have for every point of $E_0$ apart from another set of measure zero $S_1$, that $q(T - t)^\top u(t) = x(T - t)^\top p(t) = 0$. Let $E = E_0 \setminus S_1$.

At the same time, differentiating the constraints of M-CLP/M-CLP* everywhere on the set $E$ we obtain:

$$Au(t) + \dot{x}(t) = b \quad A^T p(t) - \dot{q}(t) = c$$

(23)

where $\dot{x}(t), \dot{q}(t)$ are the slopes of the corresponding slacks.

Note that on $E$, $x(t) = 0 \implies \dot{x}(t) = 0$ and $q(t) = 0 \implies \dot{q}(t) = 0$, by non-negativity. Hence, for every $t \in E$ the following holds:

$$\dot{q}(T - t)^\top u(t) = \dot{x}(T - t)^\top p(t) = 0$$

(24)

Recall also that $U(t), P(t)$ and hence also $U^*_t, P^*_t$ are non-decreasing, so:

$$u(t) \geq 0, \quad p(t) \geq 0.$$  

(25)

Consider now any point $t \in E$, and the values of $u(t), x(t), p(T - t), q(T - t)$. Let $\mathcal{J}(t), \mathcal{K}(t)$ be the indices of the non-zero components of $u(t)$ and of $p(T - t)$ respectively. One can see that (23), (24), (25) imply that $u = u(t), x = \dot{x}(t), p = p(T - t), \dot{q} = \dot{q}(T - t)$ are optimal solutions of the following pair of linear programming problems:

$$\text{max } c^\top u$$

s.t.  

$$\text{Rates-LP}(t) \quad Au + I \dot{x} = b$$

$$u_j \in \mathbb{Z} \text{ for } j \notin \mathcal{J}(t) \quad u_j \in \mathbb{P} \text{ for } j \in \mathcal{J}(t)$$

$$\dot{x}_k \in \mathbb{U} \text{ for } k \notin \mathcal{K}(t) \quad \dot{x}_k \in \mathbb{P} \text{ for } k \in \mathcal{K}(t)$$

$$\text{min } b^\top p$$

s.t.  

$$\text{Rates-LP}^*(t) \quad A^T p - I \dot{q} = c$$

$$p_k \in \mathbb{Z} \text{ for } k \notin \mathcal{K}(t) \quad p_k \in \mathbb{P} \text{ for } k \in \mathcal{K}(t)$$

$$\dot{q}_j \in \mathbb{U} \text{ for } j \notin \mathcal{J}(t) \quad \dot{q}_j \in \mathbb{P} \text{ for } j \in \mathcal{J}(t)$$

(26)

where by $\mathbb{Z}, \mathbb{P}, \mathbb{U}$ we denote following sign restrictions: $\mathbb{Z} = \{0\}$ is zero, $\mathbb{P} = \mathbb{R}_+$ is non-negative and $\mathbb{U} = \mathbb{R}$ is unrestricted.

Let $M$ be the finite number of subsets of indices $\mathcal{J}(t), \mathcal{K}(t)$ for which the dual pair of linear programs (26) is feasible. Then it follows that the values of $c^\top u(t)$ for all $t \in E$ must be the objective values of an optimal solution of (26), for one of these subsets. Since by Proposition 5.3(ii) $c^\top u(t)$ is non-decreasing there must exist a partition $0 = t_0 < t_1 < \cdots < t_N = T, N \leq M$ such that $c^\top u(t)$ is constant over each interval $(t_{n-1}, t_n) \cap E$. Recall that by Proposition 5.2 $c^\top U(t)$ is absolutely continuous on $(0, T)$, and hence $c^\top U(t) = c^\top U(t) + \int_0^t c^\top u(t)dt$. It follows that $c^\top U(t)$ is continuous piecewise linear on $(0, T)$.

(ii) Consider the following solution of M-CLP:

$$U^*(t) = \begin{cases} U(t_n) + \frac{t - t_n}{t_{n+1} - t_n} (U(t_{n+1}) - U(t_n)), & t \in (t_n, t_{n+1}], n = 0, \ldots, N - 2 \\ U(t), & t \in (t_{N-1}, t_N), \\ 0, & t = 0, T. \end{cases}$$

where $0 = t_0 < t_1 < \cdots < t_N = T$ is the time partition defined in the proof of (i). Note, that $U^*(t)$ is piecewise linear and absolutely continuous on $(0, T)$. One can see that $U^*(t)$ is also
a feasible solution for M-CLP. Let \( u^*(t) = \frac{dU^*(t)}{dt} \). Similarly to the Proposition 5.3(iii) and based on its proof we could rewrite the objective value possessed by \( U^*(t) \) by following:

\[
\int_0^T (\gamma + (T-t)c)^\top dU^*(t) = \gamma^\top U_O(T) + c^\top T U_O(T) - \int_0^T c^\top t u^*(t) dt
\]

(27)

Recall also, that by Proposition 5.2 \( c^\top U_O(t) \) is concave on \((0, T)\) and hence absolutely continuous on \((0, T)\). Then, comparing the objective values for \( U_O(t) \) and \( U^*(t) \) we obtain:

\[
\int_0^T (\gamma^\top + c^\top (T-t))dU_O(t) - \int_0^T (\gamma^\top + c^\top (T-t))dU^*(t)
\]

\[
= \int_0^T c^\top t u^*(t) dt - \int_0^T c^\top t u(t) dt
\]

\[
= \sum_{n=0}^{N-1} \int_{I_n} t \left( \frac{d}{dt} (c^\top U_O(t_n) + \frac{t-t_n}{t_{n+1} - t_n} (c^\top U_O(t_{n+1}) - c^\top U_O(t_n))) - c^\top u(t) \right) dt = 0
\]

where the first equality follows from Proposition 5.3(iii) and (27), and the second follows from (i) of this Theorem. Hence, \( U^*(t) \) is an optimal solution of M-CLP.

(iii) Let \( U_O(t) \) be any optimal solution of M-CLP. As shown in (i), there is a time partition \( 0 = t_0 < t_1 < \cdots < t_N = T \) so that \( c^\top U_O(t) \) is continuous piecewise linear, with constant slope in each interval, where the slopes in successive intervals are strictly decreasing. Let \( I_n = (t_{n-1}, t_n), n = 1, \ldots, N-1, I_N = (t_{N-1}, t_N) \). As shown in (ii) we can construct a dual optimal solution \( P^*(t) \) which is continuous piecewise linear on \((0, T)\), with breakpoints \( 0 = t_0 < t_1 < \cdots < t_N = T \). For this dual solution we have that \( p(t) = \frac{dP^*(t)}{dt} \) is constant on each interval \( I_n \), let \( p_n = p(t), t \in I_n \) denote this vector value. Then as shown in (i), \( p_n \) is a solution of the dual Rates-LP* (26).

Consider now \( u(t) = \frac{dU_O(t)}{dt} \) which is defined almost everywhere on \((0, T)\). As shown in (i), \( u(t) \) is an optimal solution of the primal Rates-LP (26), and \( u(t) \) is complementary slack to \( p(t) \) almost everywhere. By the non-degeneracy assumption 5.4, \( p_n \) is non-degenerate. Hence, \( u(t) \) is uniquely determined almost everywhere on \( I_n \), as the unique solution which is complementary slack to \( p_n \). Denote this solution by \( u_n, n = 1, \ldots, N \). This uniquely determines \( U_O(t), t \in (0, T) \), the absolutely continuous part of \( U_O(t) \), as the continuous piecewise linear vector of functions with slopes \( u_n \) in each interval.

It remains to show that \( U_O \) is absolutely continuous in \((0, T)\), i.e that \( U_s(T^-) - U_s(0^+) = 0 \).

Assume to the contrary that in some interval \((t_{m-1}, t_m] \) we have \( U_s(t_m) - U_s(t_{m-1}) > 0 \) (or if \( m = N, U_s(T^-) - U_s(t_{N-1}) > 0 \). Define

\[
U^*_m(t) = \begin{cases} 
U^*(t) & t \in I_m \\
U_O(t) & t \notin I_m
\end{cases}
\]

\[
u^*_m(t) = \begin{cases} 
\frac{dU^*(t)}{dt} & t \in I_m \\
u(t) & t \notin I_m.
\end{cases}
\]

where \( U^*(t) \) on \( I_m \) is the linear interpolation as defined in proof of (ii), and \( u^*_m = u^*_m(t) \) for \( t \in I_m \) is the constant slope \( U_O(t_m) - U_O(t_{m-1}) \) on \( I_m \).

Similar to (ii), it follows that \( U^*_m(t) \) is also an optimal solution of M-CLP. Furthermore, since the solutions are identical on \( t \notin I_m \), we must by (i) have that \( c^\top u_m = c^\top u^*_m \).

We now have, on the one hand, that:

\[
u^*_m = \frac{U_O(t_m) - U_O(t_{m-1})}{t_m - t_{m-1}} = u_m + \frac{1}{t_m - t_{m-1}} (U_s(t_m) - U_s(t_{m-1})) \neq u_m
\]
On the other hand, as we saw before, \( u^*(t) \) must be complementary slack to \( p(t) \) almost everywhere, and hence, \( u^*(t) = u_m \). This contradiction proves that \( \mathcal{U}_s(T^-) - \mathcal{U}_s(0^+) = 0 \), and shows that \( \mathcal{U}_s(t) \) is absolutely continuous on \((0, T)\).

Furthermore, \( \mathcal{U}_s(t) \) for \( t \in (0, T) \) is continuous piecewise linear, and \( \mathcal{U}_s(t) - \mathcal{U}_s(0^+) \) is uniquely determined by the partition \( 0 = t_0 < t_1 < \cdots < t_N = T \) and the slope vectors \( u_m \). This completes the proof or (iii).

\[ \square \]

REFERENCES


