

# A Two Node Jackson Network with Infinite Supply of Work

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## Abstract

We consider a Jackson network with two nodes, with no exogenous input, but instead an infinite supply of work at each of the nodes: whenever a node is empty, it processes a job from this infinite supply. We obtain an explicit expression for the steady state distribution of this system, as an infinite sum of product forms.

Keywords: Queueing, manufacturing, communication networks, Jackson networks, Markovian multiclass queueing networks, infinite virtual buffers, steady state, distributions, compensation approach.

## 1 Introduction

We consider a Jackson network with two nodes, numbered  $i = 1, 2$ . Processing times at the nodes are independent and exponentially distributed with rates  $\mu_i$ , and jobs completing processing at node  $i$  move to node  $3 - i$  with probability  $p_i$  and leave the system otherwise. There is no exogenous input to the system. However, whenever one of the nodes is empty, it will process a job from an infinite supply of jobs. This system can be described by a two dimensional Markov jump process,  $X(t) = (X_1(t), X_2(t))$ , the state space of which consists of the pairs of nonnegative integers  $(n_1, n_2)$  where  $n_1$  indicates the number of jobs at node 1, and  $n_2$  indicates the number of jobs at node 2. Whenever  $n_i > 0$  node  $i$  will process one of the jobs at the node. This introduces the transitions:

$$(n_1, n_2) \rightarrow (n_1 - 1, n_2 + 1) \text{ at rate } \mu_1 p_1, \quad n_1 > 0,$$

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$$\begin{aligned}
(n_1, n_2) &\rightarrow (n_1 - 1, n_2) \text{ at rate } \mu_1(1 - p_1), \quad n_1 > 0, \\
(n_1, n_2) &\rightarrow (n_1 + 1, n_2 - 1) \text{ at rate } \mu_2 p_2, \quad n_2 > 0, \\
(n_1, n_2) &\rightarrow (n_1, n_2 - 1) \text{ at rate } \mu_2(1 - p_2), \quad n_2 > 0.
\end{aligned} \tag{1.1}$$

Whenever node  $i$  is empty, it will process a job from its infinite supply, at the same rate  $\mu_i$ , and upon completion this job will move to the other node with probability  $p_i$ , and leave the system with probability  $1 - p_i$ . This introduces the additional transitions:

$$\begin{aligned}
(0, n_2) &\rightarrow (0, n_2 + 1) \text{ at rate } \mu_1 p_1, \\
(n_1, 0) &\rightarrow (n_1 + 1, 0) \text{ at rate } \mu_2 p_2,
\end{aligned} \tag{1.2}$$

Note that jobs from the infinite supply of each buffer are indistinguishable from jobs queued at the nodes, but queued jobs have preemptive priority over jobs in the infinite supply. The transitions (1.2) constitute arrivals into the system.

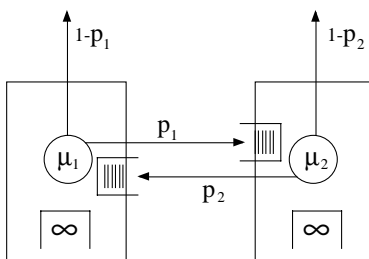


Figure 1: A two node Jackson network with infinite supply of work

The two nodes in this system are processing jobs all the time. Hence there are four independent Poisson streams in this system: Jobs depart the system in two Poisson streams with rates  $\mu_1(1 - p_1)$ ,  $\mu_2(1 - p_2)$ , and jobs arrive at the two nodes in two Poisson streams, with rates  $\mu_1 p_1$ ,  $\mu_2 p_2$ . The queue at node  $i$  therefore behaves as an  $M/M/1$  queue, with arrival rate  $\mu_{3-i} p_{3-i}$  and service rate  $\mu_i$ . The system is stable if

$$\rho_i = \frac{\mu_{3-i} p_{3-i}}{\mu_i} < 1, \quad i = 1, 2,$$

with marginal steady state distributions

$$P_i(n) = \lim_{t \rightarrow \infty} \mathbb{P}(X_i(t) = n) = (1 - \rho_i) \rho_i^n, \quad n \geq 0, \quad i = 1, 2. \tag{1.3}$$

However, the queue lengths at the two nodes in steady state are not independent; the joint steady state distribution is not product form:

$$P(n_1, n_2) = \lim_{t \rightarrow \infty} \mathbb{P}((X_1(t), X_2(t)) = (n_1, n_2)) \neq P_1(n_1) P_2(n_2), \quad n_1, n_2 \geq 0.$$

In this note we derive explicit expressions for the joint steady state distribution of the two node system. We use the compensation approach, developed by Adan *et al.* [2] to obtain an

expression which is an infinite sum of product forms.

This two node Jackson network with infinite supply of work describes quite a useful model of cooperative service by two servers: Consider jobs which require a sequence of tasks, the first task is performed by one of the servers, the remaining tasks are performed by alternating servers. Server  $i$  performs tasks at rate  $\mu_i$ , and the job then requires an additional task with probability  $p_i$ , or else it is complete and leaves the system. We assume that each of these servers has an infinite supply of jobs to start. However, each server gives preemptive priority to tasks which it received from the other server. Each server then has a queue of jobs which are ‘in process’ and the analysis of these queues tells us how much storage for WIP (work in process) is needed, and what is the cycle time of a job from first task to completion.

The concept of infinite supply of work, in contrast to the usual queueing assumption that jobs arrive randomly, is in fact very common in many systems: Whenever a server is expensive and it is desired not to keep it idle, one tries to monitor the server, and control the inputs, so that the server never runs out of work. This is the case for an expensive machine, a highly trained server, or a high performance communication link. In each case work is shunted to such servers to prevent them from idling.

As we shall see in Section 3, infinite supply Jackson nodes provide much better performance than standard Jackson nodes.

Multi-class queueing networks with infinite supplies of jobs in some of the classes, also called infinite virtual queues, were introduced by Weiss *et al.* [1, 9, 10, 12, 13, 14], see also Levy and Yechiali [11]. They represent monitored control over job arrivals, as it often exists in manufacturing and communication systems. Jackson networks are described by Jackson [7] and Kelly [8]. Weiss [14] has discussed Jackson networks with virtual infinite buffers: He has derived flow rates and stability conditions, and partial steady state distributions. This work is also closely related to the results of Goodman and Massey [5]. The analysis in the current paper provides one example of such networks, which is highly tractable.

## 2 Main theorem

The two node Jackson network with infinite supply of work is described by a Markov jump process moving on the two dimensional non-negative integer grid. The Markov process performs a two dimensional simple random walk on the positive integer grid, with transitions only to neighboring states, and with reflecting barriers on the horizontal and vertical axes. Furthermore, in the interior of the positive quadrant the random walk has no transitions to the north, the north-east, and the east directions. The transition rates for this random walk are described in Figure 2.

For such Markov jump processes it is possible to obtain a closed form expression of the steady state distribution, by the compensation method developed in the paper of Adan *et al.* [2]. The random walk in Figure 2 has the property that the transition rates at the vertical boundary  $n_1 = 0, n_2 > 0$  and the horizontal boundary  $n_1 > 0, n_2 = 0$  are projections of the

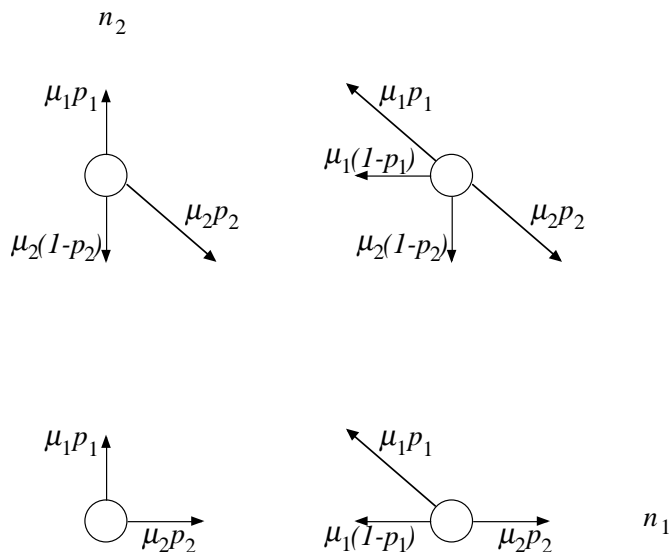


Figure 2: Transition rates for the two node system

ones in the interior  $n_1, n_2 > 0$ . Boxma and van Houtum [3] showed that this property considerably simplifies the expression of the steady-state distribution. See also [6].

The main steps in the derivation of the steady state probabilities are as follows: The balance equations for the interior are satisfied by product form expressions  $\alpha^{n_1} \beta^{n_2}$  where  $\alpha, \beta$  are solutions of a quadratic equation  $Q(\alpha, \beta) = 0$ . Solutions of this form do not as a rule satisfy the equations for the horizontal or vertical boundaries. However, it is possible to find compensating product forms such that the linear combination  $\alpha^{n_1} \beta^{n_2} + c \tilde{\alpha}^{n_1} \beta^{n_2}$  satisfies the balance equations for the interior and the vertical boundary of the quadrant. Similarly, it is possible to find compensating product forms such that  $\alpha^{n_1} \beta^{n_2} + d \alpha^{n_1} \tilde{\beta}^{n_2}$  satisfies the balance equations for the interior and the horizontal boundary of the quadrant. Then, starting with a product form  $\alpha^{n_1} \beta^{n_2}$  with  $\alpha, \beta$  satisfying  $Q(\alpha, \beta) = 0$  one can construct an infinite linear combination by adding product forms to alternately compensate for the horizontal and vertical boundary. The resulting solution formally satisfies all the balance equations. One then needs to choose the parameters of the product forms and their coefficients such that the solution is absolutely convergent. This method does indeed work for our system.

In Section 4 we will present the detailed derivation of the steady state distribution, without invoking the results in [2, 3]. In the derivation we make use of the steady state marginal distributions (1.3). This yields a particularly elegant and simple expression:

**Theorem 2.1** *The steady state distribution of the two node Jackson network with infinite*

supply of work, when  $\rho_1, \rho_2 < 1$ , is given, for all  $(n_1, n_2) \neq (0, 0)$ , by:

$$P(n_1, n_2) = \sum_{k=1}^{\infty} (-1)^{k+1} \left[ (1 - \alpha_k) \alpha_k^{n_1} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1} (1 - \beta_k) \beta_k^{n_2} \right] \quad (2.1)$$

where for  $k \geq 1$ :

$$\begin{aligned} \alpha_{k+1}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_2 p_2} \beta_k^{-1} - \alpha_{k-1}^{-1} - \frac{1 - p_2}{p_2}, \\ \beta_{k+1}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_1 p_1} \alpha_k^{-1} - \beta_{k-1}^{-1} - \frac{1 - p_1}{p_1}, \end{aligned} \quad (2.2)$$

with initially  $\alpha_0 = \beta_0 = 1, \alpha_1 = \rho_1, \beta_1 = \rho_2$ . The steady-state probability  $P(0, 0)$  is equal to:

$$P(0, 0) = 1 - \rho_1 - \rho_2 + \sum_{k=1}^{\infty} (-1)^{k+1} (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k) \quad (2.3)$$

The closed-form expression in Theorem 2.1 immediately leads to similar expressions for the distribution of the total number in the system and for the (factorial) moments of the queue lengths at node 1 and 2. Let  $X_i$  denote the queue length of node  $i$  in steady-state. Then we have:

**Corollary 2.2** (i) For all  $n > 0$ :

$$\begin{aligned} \mathbb{P}(X_1 + X_2 = n) &= \sum_{\substack{n_1 + n_2 = n \\ n_1, n_2 \geq 0}} P(n_1, n_2) = \\ &= \sum_{k=1}^{\infty} (-1)^{k+1} \left[ (1 - \alpha_k)(1 - \beta_{k+1}) \frac{\alpha_k^{n+1} - \beta_{k+1}^{n+1}}{\alpha_k - \beta_{k+1}} + (1 - \alpha_{k+1})(1 - \beta_k) \frac{\alpha_{k+1}^{n+1} - \beta_k^{n+1}}{\alpha_{k+1} - \beta_k} \right] \end{aligned} \quad (2.4)$$

(ii) For all  $m, n \geq 0, m + n > 0$ :

$$\mathbb{E} \left( \binom{X_1}{m} \binom{X_2}{n} \right) = \sum_{k=1}^{\infty} (-1)^{k+1} \left[ \frac{\alpha_k^m}{(1 - \alpha_k)^m} \frac{\beta_{k+1}^n}{(1 - \beta_{k+1})^n} + \frac{\alpha_{k+1}^m}{(1 - \alpha_{k+1})^m} \frac{\beta_k^n}{(1 - \beta_k)^n} \right]. \quad (2.5)$$

Note, exact formula for  $\alpha_k, \beta_k$  can be obtained from the difference equation (2.2), but are not particularly illuminating. The asymptotic behavior of  $\alpha_k, \beta_k$  is derived in Proposition 4.14.

### 3 Comparison with standard Jackson network

We compare our two node system with infinite supply of work and a standard Jackson network, with exogenous random inputs. Throughout this section we label our system as ‘ $\infty$ -supply’ and the Jackson network with random exogenous input as ‘standard’. We consider for the comparison two nodes in a standard Jackson network as shown in Figure 3. Here we have two

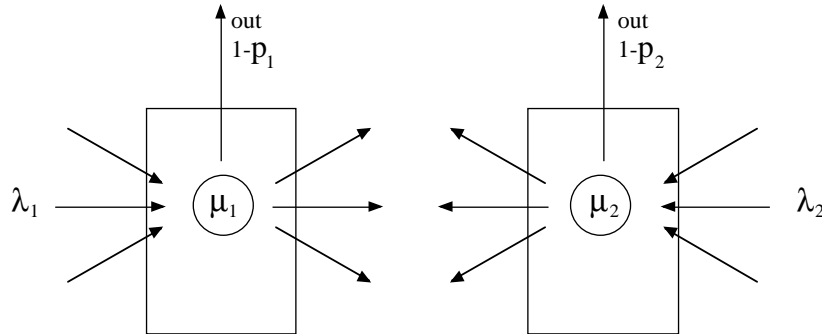


Figure 3: Standard Jackson network nodes

nodes with the same processing rates  $\mu_i$ , and with the same probabilities  $1 - p_i$  to complete a job, which then departs the system. The total inputs into the nodes are at rates  $\lambda_i$ , and they consist of both exogenous arrivals and feedback from other nodes. Recall that the *input streams are not Poisson*. The outflow in steady state is also at rate  $\lambda_i$ , and includes a *Poisson output stream of departures from the system* at rate  $\lambda_i(1 - p_i)$ . As is well known, the steady state joint distribution of the jobs in the two nodes is product form,

$$P(n_1, n_2 | \text{standard}) = \left(1 - \frac{\lambda_1}{\mu_1}\right) \left(\frac{\lambda_1}{\mu_1}\right)^{n_1} \left(1 - \frac{\lambda_2}{\mu_2}\right) \left(\frac{\lambda_2}{\mu_2}\right)^{n_2}$$

This is only stable if  $\lambda_i < \mu_i$ , and therefore the output rate of the standard nodes is always less than the rate  $(1 - p_i)\mu_i$  achieved by the  $\infty$ -supply system, and if one tries to approach this rate, the queue length explodes.

It is interesting to compare the two systems in the case that both have the same traffic intensities: For the remainder of this section we take  $\rho_i = \lambda_i/\mu_i = \mu_{3-i}p_{3-i}/\mu_i$ . We compare the total number in the two nodes for the two systems. The marginal steady state distributions in the nodes of the two systems are the same, namely Geometric, with  $P(X_i \geq n) = \rho_i^n$ . In particular it follows that the mean number in the system is the same for both networks. However the steady state distribution of the total number in the system is different.

In Figure 4 we show the distribution of the total number in the system for  $\mu_1 = 2$ ,  $\mu_2 = 3$ ,  $p_1 = 0.8$ ,  $p_2 = 0.5$  (left) and  $\mu_1 = \mu_2 = 1$ ,  $p_1 = p_2 = 0.8$  (right). We also plot the standard product form probabilities for comparison.

The correlation between  $X_1, X_2$ , calculated from the formula (2.5), equal  $-0.2976$  for the first example of an asymmetric system, and  $-0.3873$  for the second example of a symmetric system. In fact, it can be shown that the correlation is always negative, see section 4.8. Negative correlation reduces the variance of the total number in system compared to independent nodes. In Figure 5 we show the correlation between  $X_1, X_2$  for the symmetric system  $\mu_1 = \mu_2$  and  $p_1 = p_2 = p$ . Clearly, the negative correlation gets stronger as  $p$  tends to 1; the limiting value for  $p = 1$  is equal to  $\frac{2}{3}\pi^2 - 7$  (see section 4.8).

We can also get the asymptotic tail probabilities of  $X_1 + X_2$ , from (2.4). We will show that the sum is absolutely convergent and that the parameters  $\alpha_k, \beta_k$  monotonously decrease. The

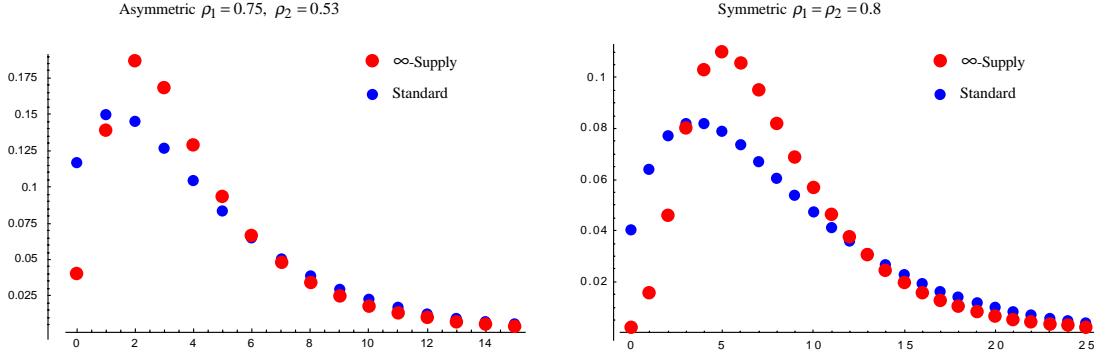


Figure 4: Probabilities of total number in the system

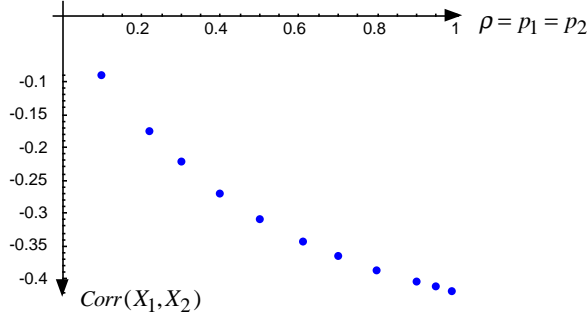


Figure 5: Correlation between  $X_1, X_2$  for the symmetric system  $\mu_1 = \mu_2$  and  $p_1 = p_2 = p$

values for large  $n$  therefore behave like the largest geometric term,

$$\mathbb{P}(X_1 + X_2 \geq n | \infty\text{-supply}) \sim \begin{cases} \frac{1-\beta_2}{1-\beta_2/\rho_1} \rho_1^n, & \rho_1 > \rho_2, \\ \left( \frac{1-\beta_2}{1-\beta_2/\rho_1} + \frac{1-\alpha_2}{1-\alpha_2/\rho_1} \right) \rho_1^n, & \rho_1 = \rho_2. \end{cases} \quad (3.1)$$

The corresponding asymptotics for the product form standard Jackson network are

$$\mathbb{P}(X_1 + X_2 \geq n | \text{standard}) \sim \begin{cases} \frac{1-\rho_2}{1-\rho_2/\rho_1} \rho_1^n, & \rho_1 > \rho_2, \\ n(1-\rho_1) \rho_1^n, & \rho_1 = \rho_2. \end{cases}$$

Hence, the asymptotic ratio of the two is:

$$\frac{\mathbb{P}(X_1 + X_2 \geq n | \text{standard})}{\mathbb{P}(X_1 + X_2 \geq n | \infty\text{-supply})} \sim \begin{cases} \frac{1-\rho_2}{1-\rho_2/\rho_1} / \frac{1-\beta_2}{1-\beta_2/\rho_1}, & \rho_1 > \rho_2, \\ \left[ (1-\rho_1) / \left( \frac{1-\beta_2}{1-\beta_2/\rho_1} + \frac{1-\alpha_2}{1-\alpha_2/\rho_1} \right) \right] n, & \rho_1 = \rho_2. \end{cases} \quad (3.2)$$

In Table 1 we summarize various quantities for the two examples, and the comparison of the total number in the system for the  $\infty$ -supply and standard system.

The most interesting part here is the strong form of variance reduction and tail probability (i.e. overflow probabilities in practice) which is obtained in the infinite supply network, when the two nodes are symmetric.

Table 1: *Comparison of Infinite Supply and Standard Jackson Networks*

		Asymmetric Example	symmetric Example
Data	$\mu_1$	2	1
	$\mu_2$	3	1
	$p_1$	0.8	0.8
	$p_2$	0.5	0.8
	$\rho_1$	0.75	0.8
	$\rho_2$	0.533	0.8
Moments	$E(X_1)$	3	4
	$E(X_2)$	1.14	4
	$V(X_1)$	12	20
	$V(X_2)$	2.45	20
	$Cov(X_1, X_2)$	-1.61	-7.75
	$Corr(X_1, X_2)$	-0.2976	-0.3873
$\frac{V(X_1+X_2 \text{standard})}{V(X_1+X_2 \infty\text{-supply})}$		1.287	1.633
$\sim \frac{P(X_1+X_2 \geq n \text{standard})}{P(X_1+X_2 \geq n \infty\text{-supply})}$		1.33445	0.07143 $n$

## 4 Derivation of steady state distribution

In this section we prove Theorem 2.1. We first derive the expression as a formal solution to the balance equations, and then prove that this solution is absolutely convergent.

### 4.1 The balance equations

The balance equations for the steady state probabilities in this system are obtained by equating the flow out and into each state, yielding:

$$\begin{aligned}
 (\mu_1 + \mu_2)P(n_1, n_2) &= \mu_1 p_1 P(n_1 + 1, n_2 - 1) + \mu_1 (1 - p_1) P(n_1 + 1, n_2) + \\
 &\quad + \mu_2 p_2 P(n_1 - 1, n_2 + 1) + \mu_2 (1 - p_2) P(n_1, n_2 + 1), \quad (4.1) \\
 &\qquad\qquad\qquad n_1, n_2 > 0,
 \end{aligned}$$

$$\begin{aligned}
 (\mu_1 + \mu_2 p_2)P(n_1, 0) &= \mu_2 p_2 P(n_1 - 1, 0) + \mu_1 (1 - p_1) P(n_1 + 1, 0) \\
 &\quad + \mu_2 p_2 P(n_1 - 1, 1) + \mu_2 (1 - p_2) P(n_1, 1), \quad n_1 > 0, \quad (4.2)
 \end{aligned}$$

$$\begin{aligned}
 (\mu_1 p_1 + \mu_2)P(0, n_2) &= \mu_1 p_1 P(0, n_2 - 1) + \mu_2 (1 - p_2) P(0, n_2 + 1) \\
 &\quad + \mu_1 p_1 P(1, n_2 - 1) + \mu_1 (1 - p_1) P(1, n_2), \quad n_2 > 0, \quad (4.3)
 \end{aligned}$$

$$(\mu_1 p_1 + \mu_2 p_2)P(0, 0) = \mu_1 (1 - p_1) P(1, 0) + \mu_2 (1 - p_2) P(0, 1). \quad (4.4)$$

In the next section we will characterize the product forms  $\alpha^{n_1} \beta^{n_2}$  satisfying the balance equations in the interior of the quadrant.

## 4.2 Product form trial solutions in the interior of the quadrant

Consider first the equations (4.1) in the interior of the quadrant, and a product form trial solution  $\alpha^{n_1}\beta^{n_2}$ . Substituting this trial solution in (4.1) and canceling  $\alpha^{n_1-1}\beta^{n_2-1}$  we see immediately that:

**Proposition 4.1** *The product form  $\alpha^{n_1}\beta^{n_2}$  solves the equation (4.1) for every  $n_1, n_2 = 0, \pm 1, \pm 2, \dots$ , if and only if  $\alpha, \beta$  are on the curve:*

$$(\mu_1 + \mu_2)\alpha\beta = \alpha^2(\mu_1 p_1 + \mu_1(1 - p_1)\beta) + \beta^2(\mu_2 p_2 + \mu_2(1 - p_2)\alpha). \quad (4.5)$$

The curve (4.5) is shown in Figure 6. The pairs of values  $(\alpha, \beta) = (0, 0)$ , and  $(\alpha, \beta) = (1, 1)$ , are on this curve. We also illustrate on the curve how the special roots which appear in the solution (2.1,2.2),  $\alpha_k, \beta_k$ , are calculated, for  $k = 0, 1, 2, 3$ .

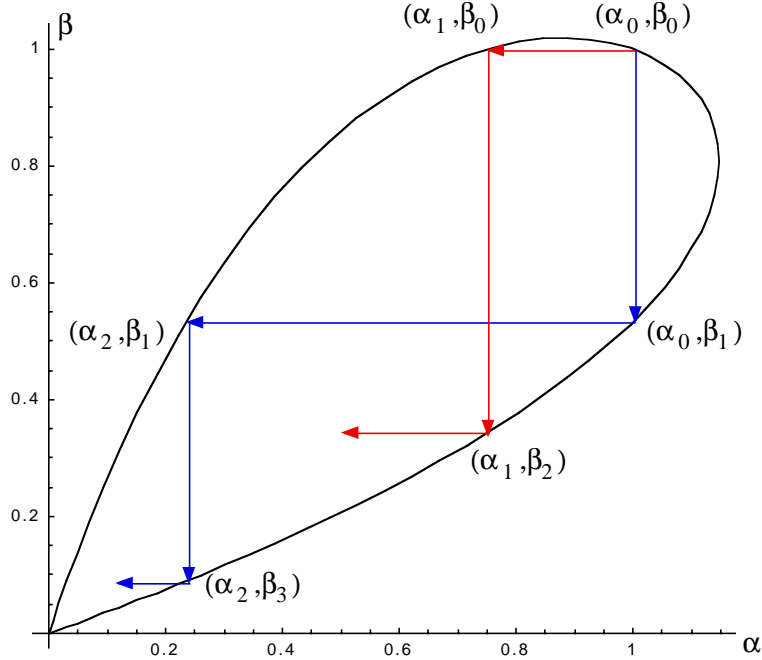


Figure 6: Curve (4.5) for  $\mu_1 = 2, \mu_2 = 3$  and  $p_1 = 0.8, p_2 = 0.5$

For every fixed value of  $0 < \alpha \leq 1$ , equation (4.5) yields a quadratic equation for  $\beta$ :

$$[\mu_2 p_2 + \mu_2(1 - p_2)\alpha] \beta^2 - [(\mu_1 + \mu_2)\alpha - \mu_1(1 - p_1)\alpha^2] \beta + [\mu_1 p_1 \alpha^2] = 0. \quad (4.6)$$

**Proposition 4.2** *The quadratic equation (4.6) has two real roots for all  $0 < \alpha \leq 1$ . For  $\alpha = 1$  the roots are  $\bar{\beta} = 1$ ,  $\underline{\beta} = \rho_2$ . For  $0 < \alpha < 1$  the larger root is  $\bar{\beta} > \alpha$ , and the smaller root is  $0 < \underline{\beta} < \alpha$ .*

**Proof.** For the fixed value  $\alpha = \alpha_0 = 1$  the quadratic equation (4.6) for  $\beta$  is

$$\mu_2\beta^2 - (\mu_1p_1 + \mu_2)\beta + \mu_1p_1 = 0,$$

with the two roots  $\bar{\beta} = 1$  and  $\underline{\beta} = \beta_1 = \mu_1p_1/\mu_2 = \rho_2$ . For  $0 < \alpha < 1$ , if we substitute  $\beta = \alpha$  in the right-hand side of the quadratic equation (4.6) we get:

$$\alpha^2(\alpha - 1)(\mu_1(1 - p_1) + \mu_2(1 - p_2)) < 0.$$

Hence the quadratic equation (4.6) has two roots, one of them larger and the other smaller than  $\alpha$ . The product of the two roots is  $\mu_1p_1/\mu_2$ , hence both are positive. ■

Similarly, for every fixed value  $0 < \beta \leq 1$ , equation (4.5) yields a quadratic equation for  $\alpha$ :

$$[\mu_1p_1 + \mu_1(1 - p_1)\beta] \alpha^2 - [(\mu_1 + \mu_2)\beta - \mu_2(1 - p_2)\beta^2] \alpha + [\mu_2p_2\beta^2] = 0, \quad (4.7)$$

**Proposition 4.3** *The quadratic equation (4.7) has two real roots for all  $0 < \beta \leq 1$ . For  $\beta = 1$  the roots are  $\bar{\alpha} = 1$ ,  $\underline{\alpha} = \rho_1$ . For  $0 < \beta < 1$  the larger root is  $\bar{\alpha} > \beta$  the smaller root is  $0 < \underline{\alpha} < \beta$ .*

It is convenient to divide the quadratic equations (4.6, 4.7) by  $\alpha^2\beta^2$ , and to consider quadratic equations for  $\alpha^{-1}$ ,  $\beta^{-1}$ :

$$\begin{aligned} [\mu_1p_1] (\beta^{-1})^2 &- [(\mu_1 + \mu_2)\alpha^{-1} - \mu_1(1 - p_1)] (\beta^{-1}) + \\ &+ [(\alpha^{-1}) (\mu_2(1 - p_2) + \mu_2p_2\alpha^{-1})] &= 0 \end{aligned} \quad (4.8)$$

$$\begin{aligned} [\mu_2p_2] (\alpha^{-1})^2 &- [(\mu_1 + \mu_2)\beta^{-1} - \mu_2(1 - p_2)] (\alpha^{-1}) + \\ &+ [(\beta^{-1}) (\mu_1(1 - p_1) + \mu_1p_1\beta^{-1})] &= 0. \end{aligned} \quad (4.9)$$

### 4.3 Compensating for the vertical and for the horizontal boundary

Let  $\alpha, \beta$  satisfy (4.5), so that  $\alpha^{n_1}\beta^{n_2}$  solve the balance equations (4.1), for all  $n_1, n_2 = 0, \pm 1, \pm 2, \dots$ . We want to find a compensating term such that  $\alpha^{n_1}\beta^{n_2} + c\tilde{\alpha}^{n_1}\tilde{\beta}^{n_2}$  will in addition also solve the horizontal boundary equations (4.2).

We first subtract the equation (4.2) from (4.1), to obtain the equation

$$\mu_2(1 - p_2)P(n_1, 0) + \mu_2p_2P(n_1 - 1, 0) = \mu_1p_1P(n_1 + 1, -1), \quad n_1 > 0. \quad (4.10)$$

Since our trial solution  $\alpha^{n_1}\beta^{n_2} + c\tilde{\alpha}^{n_1}\tilde{\beta}^{n_2}$  solves (4.1), it will solve (4.2) if and only if it solves (4.10). We substitute the trial solution in (4.10), yielding

$$[\mu_2(1 - p_2) + \mu_2p_2\alpha^{-1} - \mu_1p_1\alpha\beta^{-1}]\alpha^{n_1} + c[\mu_2(1 - p_2) + \mu_2p_2\tilde{\alpha}^{-1} - \mu_1p_1\tilde{\alpha}\tilde{\beta}^{-1}]\tilde{\alpha}^{n_1} = 0. \quad (4.11)$$

To satisfy (4.11) for all  $n_1 > 0$  we are forced to take  $\tilde{\alpha} = \alpha$  and thus, to solve (4.1) we need to take  $\tilde{\beta}$  as the second root of the quadratic equation (4.6). Using the quadratic equation (4.8) we get the second root  $\tilde{\beta}^{-1}$  in terms of  $\alpha$  and the first root  $\beta^{-1}$ :

$$\tilde{\beta}^{-1} = \frac{\mu_1 + \mu_2}{\mu_1p_1}\alpha^{-1} - \beta^{-1} - \frac{1 - p_1}{p_1}.$$

We also get the product of the roots of (4.8):

$$\mu_1 p_1 \beta^{-1} \tilde{\beta}^{-1} = \alpha^{-1} (\mu_2 (1 - p_2) + \mu_2 p_2 \alpha^{-1}). \quad (4.12)$$

By canceling  $\alpha^{n_1+1}$  in (4.11) we obtain an equation for  $c$ :

$$(1 + c) \alpha^{-1} (\mu_2 (1 - p_2) + \mu_2 p_2 \alpha^{-1}) = \mu_1 p_1 (\beta^{-1} + c \tilde{\beta}^{-1}) \quad (4.13)$$

We now use (4.12) to cancel  $\mu_1 p_1 \beta^{-1} \tilde{\beta}^{-1}$  on both sides, and obtain:

$$c = -\frac{1 - \tilde{\beta}}{1 - \beta}$$

Multiplying the linear combination by the constant  $(1 - \alpha)(1 - \beta)$ , we may conclude that  $(1 - \alpha) \alpha^{n_1} (1 - \beta) \beta^{n_2} - (1 - \alpha) \alpha^{n_1} (1 - \tilde{\beta}) \tilde{\beta}^{n_2}$  solves the balance equations (4.1, 4.2). The procedure to compensate for the vertical boundary equations is symmetric.

**Proposition 4.4** *Let  $\alpha, \beta$  satisfy (4.5). Then  $(1 - \alpha) \alpha^{n_1} (1 - \beta) \beta^{n_2} - (1 - \alpha) \alpha^{n_1} (1 - \tilde{\beta}) \tilde{\beta}^{n_2}$  solves the balance equations (4.1, 4.2) in the interior and the horizontal boundary if we take:*

$$\tilde{\beta}^{-1} = \frac{\mu_1 + \mu_2}{\mu_1 p_1} \alpha^{-1} - \beta^{-1} - \frac{1 - p_1}{p_1} \quad (4.14)$$

Similarly,  $(1 - \alpha) \alpha^{n_1} (1 - \beta) \beta^{n_2} + (1 - \tilde{\alpha}) \tilde{\alpha}^{n_1} (1 - \beta) \beta^{n_2}$  solves the balance equations (4.1, 4.3) in the interior and the vertical boundary if we take:

$$\tilde{\alpha}^{-1} = \frac{\mu_1 + \mu_2}{\mu_2 p_2} \beta^{-1} - \alpha^{-1} - \frac{1 - p_2}{p_2}. \quad (4.15)$$

#### 4.4 Infinite sequences of compensations

Motivated by the marginal distribution (1.3) we start from a product form solution with  $\alpha_1 = \rho_1$ . The roots of (4.6) are  $\beta_0 = 1$ , and  $\beta_2 < \rho_1$ . Since we need convergence we start from the trial solution  $\alpha_1^{n_1} \beta_2^{n_2}$ . To conform with our desired final form we multiply this trial solution by a constant:

**Proposition 4.5** *The trial solution  $(1 - \alpha_1) \alpha_1^{n_1} (1 - \beta_2) \beta_2^{n_2}$  with  $\alpha_1 = \rho_1$  and  $\beta_2^{-1} = \frac{\mu_1 + \mu_2}{\mu_1 p_1} \rho_1^{-1} - 1 - \frac{1 - p_1}{p_1}$  solves the equations (4.1, 4.2) for all  $n_1 > 0, n_2 \geq 0$ .*

**Proof.** In this case the compensating term would have  $\tilde{\beta} = 1$ , but then  $1 - \tilde{\beta} = 0$ , so the compensating term disappears. ■

We next add a compensating term to solve (4.1, 4.3). According to (4.15) we choose

$$\alpha_3^{-1} = \frac{\mu_1 + \mu_2}{\mu_2 p_2} \beta_2^{-1} - \alpha_1^{-1} - \frac{1 - p_2}{p_2}$$

to obtain a two term trial solution

$$(1 - \alpha_1) \alpha_1^{n_1} (1 - \beta_2) \beta_2^{n_2} - (1 - \alpha_3) \alpha_3^{n_1} (1 - \beta_2) \beta_2^{n_2}. \quad (4.16)$$

In this solution the first term alone solves (4.1, 4.2), and the two terms together solve (4.1, 4.3).

From now on we continue to add compensating terms, to satisfy (4.2) and to satisfy (4.3) alternately. In the next step we need to compensate the second term of (4.16) to solve (4.2) again, and we choose  $\beta_4$  according to (4.14). We continue these compensating steps indefinitely.

**Proposition 4.6** *Let for all  $k \geq 1$ ,*

$$\begin{aligned}\beta_{2k}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_1 p_1} \alpha_{2k-1}^{-1} - \beta_{2k-2}^{-1} - \frac{1-p_1}{p_1}, \\ \alpha_{2k+1}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_2 p_2} \beta_{2k}^{-1} - \alpha_{2k-1}^{-1} - \frac{1-p_2}{p_2},\end{aligned}\tag{4.17}$$

with initially  $\beta_0 = 1$ ,  $\alpha_1 = \rho_1$ . Then the following trial solution

$$\sum_{k=1}^{\infty} [(1 - \alpha_{2k-1}) \alpha_{2k-1}^{n_1} (1 - \beta_{2k}) \beta_{2k}^{n_2} - (1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k}) \beta_{2k}^{n_2}] \tag{4.18}$$

solves the balance equations for all  $(n_1, n_2) \neq (0, 1), (1, 0), (0, 0)$ .

**Proof.** We show in section 4.7 that the infinite sum (4.18) is absolutely convergent for every  $(n_1, n_2) \neq (0, 0)$ . We will also show that the summation of (4.18) over all the values of  $(n_1, n_2) \neq (0, 0)$  converges absolutely. In the rest of the proof we take this statement as proved.

The pair  $(\alpha_1, \beta_0)$  is on the curve (4.5). Hence, using (4.14, 4.15) and induction, so are all the pairs  $(\alpha_{2k-1}, \beta_{2k})$  and  $(\alpha_{2k+1}, \beta_{2k})$ . Hence all the terms in (4.18) solve (4.1), and by absolute convergence so does the infinite sum for  $n_1 > 0, n_2 > 0$ .

In the sum (4.18) each negative term compensates the preceding positive term so that their sum solve (4.3); see Proposition 4.4. Hence, for all  $K$ ,

$$\sum_{k=1}^K [(1 - \alpha_{2k-1}) \alpha_{2k-1}^{n_1} (1 - \beta_{2k}) \beta_{2k}^{n_2} - (1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k}) \beta_{2k}^{n_2}]$$

solves (4.3). By absolute convergence (4.18) solves (4.3), whenever the equations do not involve  $(n_1, n_2) = (0, 0)$ . Hence, (4.18) solves (4.3) for all  $(0, n_2), n_2 > 1$ .

We saw that  $(1 - \alpha_1) \alpha_1^{n_1} (1 - \beta_2) \beta_2^{n_2}$  solves (4.2). Each positive term  $(1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k+2}) \beta_{2k+2}^{n_2}$  compensates the preceding negative term  $-(1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k}) \beta_{2k}^{n_2}$  in the sum, so that their sum solves (4.2). Hence, for all  $K$ ,

$$\begin{aligned}(1 - \alpha_1) \alpha_1^{n_1} (1 - \beta_2) \beta_2^{n_2} + \\ \sum_{k=1}^K [-(1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k}) \beta_{2k}^{n_2} + (1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k+2}) \beta_{2k+2}^{n_2}]\end{aligned}$$

solves (4.2). By absolute convergence (4.18) solves (4.2), whenever the equations do not involve  $(n_1, n_2) = (0, 0)$ . Hence, (4.18) solves (4.2) for all  $(n_1, 0), n_1 > 1$ . ■

Analogously we can start from the  $(1, \rho_2)$  on the curve (4.5), and get another solution:

**Proposition 4.7** *Let for all  $k \geq 1$ ,*

$$\begin{aligned}\alpha_{2k}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_2 p_2} \beta_{2k-1}^{-1} - \alpha_{2k-2}^{-1} - \frac{1 - p_2}{p_2}, \\ \beta_{2k+1}^{-1} &= \frac{\mu_1 + \mu_2}{\mu_1 p_1} \alpha_{2k}^{-1} - \beta_{2k-1}^{-1} - \frac{1 - p_1}{p_1}.\end{aligned}\tag{4.19}$$

*with initially  $\alpha_0 = 1, \beta_1 = p_2$ . Then the following trial solution*

$$\sum_{k=1}^{\infty} [(1 - \alpha_{2k}) \alpha_{2k}^{n_1} (1 - \beta_{2k-1}) \beta_{2k-1}^{n_2} - (1 - \alpha_{2k}) \alpha_{2k}^{n_1} (1 - \beta_{2k+1}) \beta_{2k+1}^{n_2}] \tag{4.20}$$

*solves the balance equations for all  $(n_1, n_2) \neq (0, 1), (1, 0), (0, 0)$ .*

#### 4.5 The complete solution

The two solutions in Propositions 4.6, 4.7 were not defined for  $(n_1, n_2) = (0, 0)$ . The reason is that for  $n_1 = n_2 = 0$  the sums are not (absolutely) convergent, and so they are meaningless. As a result we could not check for  $(n_1, n_2) = (1, 0)$  or  $(n_1, n_2) = (0, 1)$ .

To obtain a solution for all  $(n_1, n_2)$  we do the following: For all  $(n_1, n_2) \neq (0, 0)$  we take the sum of the two solutions (4.18, 4.20). This yields:

$$P(n_1, n_2) = \sum_{k=1}^{\infty} (-1)^{k+1} [(1 - \alpha_k) \alpha_k^{n_1} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1} (1 - \beta_k) \beta_k^{n_2}] \tag{4.21}$$

For  $(n_1, n_2) = (0, 0)$  we take:

$$P(0, 0) = 1 - \alpha_1 - \beta_1 + \sum_{k=1}^{\infty} (-1)^{k+1} (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k). \tag{4.22}$$

**Proposition 4.8** *The expressions for  $P(n_1, n_2)$  in (4.21, 4.22) solve all the balance equations.*

**Proof.** We shall show in section 4.7 that the sum in  $P(0, 0)$  is also absolutely convergent. We shall take that as well as absolute convergence of all the other  $P(n_1, n_2)$ , and their sum over all  $n_1, n_2$  as proved.

We already know by the previous two propositions that  $P(n_1, n_2)$  defined by (4.21) satisfy the balance equations (4.1, 4.2, 4.3) for  $n_1 + n_2 > 1$ . It remains to consider the balance equations for  $(1, 0), (0, 1), (0, 0)$ . For all  $K$  we have seen in the proof of Proposition 4.6 that the sum

$$\begin{aligned}(1 - \alpha_1) \alpha_1^{n_1} (1 - \beta_2) \beta_2^{n_2} + \\ + \sum_{k=1}^K [-(1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k}) \beta_{2k}^{n_2} + (1 - \alpha_{2k+1}) \alpha_{2k+1}^{n_1} (1 - \beta_{2k+2}) \beta_{2k+2}^{n_2}] \end{aligned} \tag{4.23}$$

solves (4.2). Also for all  $K$  we have seen in the proof of Proposition 4.7 that the sum

$$\sum_{k=1}^K [(1 - \alpha_{2k})\alpha_{2k}^{n_1}(1 - \beta_{2k-1})\beta_{2k-1}^{n_2} - (1 - \alpha_{2k})\alpha_{2k}^{n_1}(1 - \beta_{2k+1})\beta_{2k+1}^{n_2}] \quad (4.24)$$

solves (4.2). Hence the sum of (4.23, 4.24) also solves (4.2). Consider in particular the balance equation for  $(n_1, n_2) = (1, 0)$ :

$$(\mu_1 + \mu_2 p_2)P(1, 0) = \mu_2 p_2 P(0, 0) + \mu_1(1 - p_1)P(2, 0) + \mu_2 p_2 P(0, 1) + \mu_2(1 - p_2)P(1, 1). \quad (4.25)$$

It is satisfied by the sum of (4.23, 4.24). We now look at the sum (4.23, 4.24) for  $n_1 = n_2 = 0$ :

$$\begin{aligned} & (1 - \alpha_1)(1 - \beta_2) + \\ & + \sum_{k=1}^K [-(1 - \alpha_{2k+1})(1 - \beta_{2k}) + (1 - \alpha_{2k+1})(1 - \beta_{2k+2})] + \\ & + \sum_{k=1}^K [(1 - \alpha_{2k})(1 - \beta_{2k-1}) - (1 - \alpha_{2k})(1 - \beta_{2k+1})] = \\ & = \sum_{k=1}^{2K} (-1)^{k+1} [(1 - \alpha_k)(1 - \beta_{k+1}) + (1 - \alpha_{k+1})(1 - \beta_k)] + \\ & + (1 - \alpha_{2K+1})(1 - \beta_{2K+2}) = \\ & = 1 - \alpha_1 - \beta_1 + \sum_{k=1}^{2K} (-1)^{k+1} (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k) + \\ & - \beta_{2K+2} + \alpha_{2K+1} \beta_{2K+2}. \end{aligned}$$

As we shall see,  $\alpha_k, \beta_k \rightarrow 0$  as  $k \rightarrow \infty$ . This property and absolute convergence of the sum  $\sum_{k=1}^{\infty} (-1)^{k+1} (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k)$  shows that equation (4.25) is satisfied by  $P(1, 0)$ ,  $P(0, 1)$ ,  $P(1, 1)$ ,  $P(2, 0)$  as defined in (4.21) and  $P(0, 0)$  as defined by (4.22). The proof for the balance equation of  $(0, 1)$  is symmetric.

Finally, by the absolute convergence of the sum over all  $n_1, n_2$  of (4.21), we get that the equation (4.4) is redundant, and is satisfied by (4.21, 4.22) automatically. ■

## 4.6 Normalizing the sum of probabilities

We again take absolute convergence as proved. Based on that we can calculate various quantities. We first obtain marginal probabilities, which are consistent with (1.3).

**Proposition 4.9** For  $n_i > 0$ ,

$$\sum_{n_3-i=0}^{\infty} P(n_1, n_2) = (1 - \rho_i) \rho_i^{n_i}.$$

**Proof.** We make heavy use of the absolute convergence to change order of summations and group sums of positive and negative terms. For  $n_1 > 0$  the sum is:

$$\begin{aligned}
& \sum_{n_2=0}^{\infty} P(n_1, n_2) = \\
&= \sum_{n_2=0}^{\infty} \sum_{k=1}^{\infty} (-1)^{k+1} [(1 - \alpha_k) \alpha_k^{n_1} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1} (1 - \beta_k) \beta_k^{n_2}] = \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} \left[ (1 - \alpha_k) \alpha_k^{n_1} \sum_{n_2=0}^{\infty} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1} \sum_{n_2=0}^{\infty} (1 - \beta_k) \beta_k^{n_2} \right] = \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} [(1 - \alpha_k) \alpha_k^{n_1} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1}] = \\
&= (1 - \alpha_1) \alpha_1^{n_1} - (1 - \alpha_2) \alpha_2^{n_1} + \\
&\quad + (1 - \alpha_2) \alpha_2^{n_1} - (1 - \alpha_3) \alpha_3^{n_1} + \\
&\quad \quad \quad \vdots \\
&= (1 - \alpha_1) \alpha_1^{n_1}
\end{aligned}$$

The case of  $i = 2$  is symmetric. ■

We next calculate  $P(n_1 = 0, n_2 > 0)$  and  $P(n_1 > 0, n_2 = 0)$ .

**Proposition 4.10**

$$\begin{aligned}
\sum_{n_2=1}^{\infty} P(0, n_2) &= \rho_2 + \sum_{k=1}^{\infty} (-1)^k (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k), \\
\sum_{n_1=1}^{\infty} P(n_1, 0) &= \rho_1 + \sum_{k=1}^{\infty} (-1)^k (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k).
\end{aligned}$$

**Proof.** We again make heavy use of the absolute convergence.

$$\begin{aligned}
& \sum_{n_2=1}^{\infty} P(0, n_2) = \\
&= \sum_{n_2=1}^{\infty} \sum_{k=1}^{\infty} (-1)^{k+1} [(1 - \alpha_k) (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) (1 - \beta_k) \beta_k^{n_2}] = \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} \left[ (1 - \alpha_k) \sum_{n_2=1}^{\infty} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \sum_{n_2=1}^{\infty} (1 - \beta_k) \beta_k^{n_2} \right] = \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} [(1 - \alpha_k) \beta_{k+1} + (1 - \alpha_{k+1}) \beta_k] = \\
&= \sum_{k=1}^{\infty} (-1)^{k+1} (\beta_k + \beta_{k+1}) + \sum_{k=1}^{\infty} (-1)^k (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k) = \\
&= \beta_1 + \sum_{k=1}^{\infty} (-1)^k (\alpha_k \beta_{k+1} + \alpha_{k+1} \beta_k)
\end{aligned}$$

The other case is symmetric. ■

Finally we have:

**Proposition 4.11** *The probabilities  $P(n_1, n_2)$  in (4.21, 4.22) sum up to 1.*

**Proof.** By the previous two propositions and (4.22),

$$\sum_{n_1, n_2} P(n_1, n_2) = \sum_{n_1=1}^{\infty} \sum_{n_2=0}^{\infty} P(n_1, n_2) + \sum_{n_2=1}^{\infty} P(0, n_2) + P(0, 0) = 1.$$

■

#### 4.7 Absolute convergence

In the previous sections we made heavy use of the absolute convergence of the sums in (4.21, 4.22). This will be proved below.

**Proposition 4.12**

$$\sum_{(n_1, n_2) \neq (0, 0)} \sum_{k=1}^{\infty} [(1 - \alpha_k) \alpha_k^{n_1} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1} (1 - \beta_k) \beta_k^{n_2}] < \infty$$

**Proof.**

$$\begin{aligned} & \sum_{(n_1, n_2) \neq (0, 0)} \sum_{k=1}^{\infty} [(1 - \alpha_k) \alpha_k^{n_1} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1} (1 - \beta_k) \beta_k^{n_2}] = \\ &= \sum_{k=1}^{\infty} \left[ (1 - \alpha_k) \sum_{n_2=1}^{\infty} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \sum_{n_2=1}^{\infty} (1 - \beta_k) \beta_k^{n_2} \right] + \\ & \quad + \sum_{k=1}^{\infty} \sum_{n_1=1}^{\infty} \left[ (1 - \alpha_k) \alpha_k^{n_1} \sum_{n_2=0}^{\infty} (1 - \beta_{k+1}) \beta_{k+1}^{n_2} + (1 - \alpha_{k+1}) \alpha_{k+1}^{n_1} \sum_{n_2=0}^{\infty} (1 - \beta_k) \beta_k^{n_2} \right] = \\ &= \sum_{k=1}^{\infty} ((1 - \alpha_k) \beta_{k+1} + (1 - \alpha_{k+1}) \beta_k) + \sum_{k=1}^{\infty} (\alpha_k + \alpha_{k+1}) < \\ &< 2 \sum_{k=1}^{\infty} (\alpha_k + \beta_k). \end{aligned}$$

In the next proposition we show that the sequences  $\alpha_k, \beta_k$  decrease geometrically, and hence the last sum converges. ■

**Proposition 4.13** *For all  $k \geq 0$ ,*

$$\alpha_{k+1} \leq \rho_1 \beta_k, \quad \beta_{k+1} \leq \rho_2 \alpha_k.$$

**Proof.** For  $k = 0$  we have  $\alpha_0 = \beta_0 = 1, \alpha_1 = \rho_1, \beta_1 = \rho_2$  and this is the only case of equality. For  $k \geq 1$ , by (4.15):

$$\alpha_{k+1}^{-1} = (p_2^{-1} + \rho_1^{-1}) \beta_k^{-1} - \alpha_{k-1}^{-1} - \frac{1-p_2}{p_2} > (p_2^{-1} + \rho_1^{-1} - 1) \beta_k^{-1} - p_2^{-1} + 1 > \rho_1^{-1} \beta_k^{-1},$$

where the first inequality follows from  $\beta_k^{-1} > \alpha_{k-1}^{-1}$  and the second from  $\beta_k^{-1} > 1$ . The proof for  $\beta_{k+1}$  is symmetric. ■

We can also get the asymptotic rate of decay of  $\alpha_k, \beta_k$ :

**Proposition 4.14** *As  $k \rightarrow \infty$ ,*

$$\frac{\alpha_{k+1}}{\alpha_{k-1}}, \frac{\beta_{k+1}}{\beta_{k-1}} \rightarrow \frac{\mu_1 + \mu_2 - \sqrt{(\mu_1 + \mu_2)^2 - 4\mu_1\mu_2p_1p_2}}{\mu_1 + \mu_2 + \sqrt{(\mu_1 + \mu_2)^2 - 4\mu_1\mu_2p_1p_2}}.$$

**Proof.** The parameters  $\alpha_{k+1}$  and  $\alpha_{k-1}$  are the roots of (4.5) with  $\beta = \beta_k$ . Dividing (4.5) by  $\beta_k^2$  we get that  $\alpha_{k+1}/\beta_k$  and  $\alpha_{k-1}/\beta_k$  are the roots of

$$[\mu_1p_1 + \mu_1(1-p_1)\beta]\gamma^2 - [(\mu_1 + \mu_2) - \mu_2(1-p_2)\beta]\gamma + \mu_2p_2 = 0, \quad (4.26)$$

with  $\beta = \beta_k$ . As  $k \rightarrow \infty$ , then  $\beta_k \rightarrow 0$  by Proposition 4.13 and thus

$$\frac{\alpha_{k+1}}{\beta_k} \rightarrow \gamma_1, \quad \frac{\alpha_{k-1}}{\beta_k} \rightarrow \gamma_2,$$

where  $0 < \gamma_1 < 1 < \gamma_2$  are the roots of (4.26) with  $\beta = 0$ . Hence,

$$\frac{\alpha_{k+1}}{\alpha_{k-1}} \rightarrow \frac{\gamma_1}{\gamma_2} = \frac{\mu_1 + \mu_2 - \sqrt{(\mu_1 + \mu_2)^2 - 4\mu_1\mu_2p_1p_2}}{\mu_1 + \mu_2 + \sqrt{(\mu_1 + \mu_2)^2 - 4\mu_1\mu_2p_1p_2}}.$$

The proof for  $\beta_{k+1}/\beta_k$  is similar. ■

From the geometric decay of the sequences  $\alpha_k, \beta_k$  we can further conclude:

**Corollary 4.15** (i) *The sum which defines  $P(0,0)$  in (4.22) is absolutely convergent.*

(ii) *For all  $m, n \geq 0, m+n > 0$  the sum defining  $\mathbb{E} \left( \binom{X_1}{m} \binom{X_2}{n} \right)$  in (2.5) is absolutely convergent.*

## 4.8 Non-negativity and ergodicity

From Propositions 4.8, 4.11, 4.12 it follows that  $P(n_1, n_2)$  given by (4.21, 4.22) are a nonnull, absolutely convergent solution of the balance equations, which sums up to 1. From Theorem 1 in Foster [4] we can immediately conclude that:

**Corollary 4.16** *The Markov jump process  $X(t) = (X_1(t), X_2(t))$  is ergodic when  $\rho_1, \rho_2 < 1$ , and its equilibrium probabilities are given by the solution  $P(n_1, n_2)$  defined by (4.21).*

## 4.9 Queue length correlation

In this section we show that the correlation between  $X_1$  and  $X_2$  is always negative, which is equivalent to

$$\begin{aligned} \mathbb{E}(X_1 X_2) &= \sum_{k=1}^{\infty} (-1)^{k+1} \left[ \frac{\alpha_k}{(1-\alpha_k)} \frac{\beta_{k+1}}{(1-\beta_{k+1})} + \frac{\alpha_{k+1}}{(1-\alpha_{k+1})} \frac{\beta_k}{(1-\beta_k)} \right] \\ &< \frac{\rho_1}{1-\rho_1} \frac{\rho_2}{1-\rho_2}. \end{aligned} \quad (4.27)$$

The terms in the infinite sum (4.27) are alternating and decreasing in magnitude, since  $\alpha_1 > \beta_2 > \alpha_3 > \dots$  and  $\beta_1 > \alpha_2 > \beta_3 > \dots$ . Hence, it suffices to show that

$$\frac{\alpha_1}{(1-\alpha_1)} \frac{\beta_2}{(1-\beta_2)} + \frac{\alpha_2}{(1-\alpha_2)} \frac{\beta_1}{(1-\beta_1)} < \frac{\rho_1}{1-\rho_1} \frac{\rho_2}{1-\rho_2},$$

which can be verified by straightforward calculations. This proves:

**Proposition 4.17** *If  $\rho_1, \rho_2 < 1$ , then  $\text{Corr}(X_1, X_2) < 0$ .*

Figure 5 displays the correlation for the symmetric system  $\mu_1 = \mu_2 = \mu$  and  $p_1 = p_2 = p$ . To find the limiting value of the correlation as  $p \uparrow 1$ , note that in the symmetric case,

$$\alpha_k = \beta_k = 1 - \frac{1}{2}k(k+1)(1-p) + O(1-p)^2,$$

which can be derived from the recursive relations for the sequences  $\alpha_k, \beta_k$ . Hence, from (4.27) and using that

$$\mathbb{E}(X_1) = \mathbb{E}(X_2) = \frac{p}{1-p}, \quad V(X_1) = V(X_2) = \frac{p}{(1-p)^2},$$

we obtain

$$\lim_{p \uparrow 1} \text{Corr}(X_1, X_2) = \lim_{p \uparrow 1} \frac{\mathbb{E}(X_1 X_2) - \mathbb{E}(X_1)\mathbb{E}(X_2)}{\sqrt{V(X_1)V(X_2)}} = 8 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(k+1)^2(k+2)} - 1 = \frac{2}{3}\pi^2 - 7.$$

Note, exactly the same asymptotic correlation value appears in the calculations of Boxma and van Houtum [3], page 488, which is curious, since the two models are quite different.

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