Control of queueing networks in the large deviations limit

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I. Introduction and Motivation

Control of queueing networks - asymptotic analysis
Real-life systems are becoming large, fast, complex. Optimization becomes critical, Unusual events dominate performance Yet exact results are scarce.

Our approach - asymptotic analysis as system becomes faster. We treat Markovian queues. In previous work (Atar, Dupuis, S) - the criterion was - length of exit time for a set. Here we consider a more general risk-sensitive criterion.
Parallel server model

$I$ customer classes, $J$ service stations,
$N_j \geq 1$ identical servers in station $j \in \mathcal{J} := \{1, 2, \ldots, J\}$.
Arrivals $E_i, i \in \mathcal{I} := \{1, 2, \ldots, I\}$ are Poisson $\lambda_i$
Service of customer $i$ by server $j$ is exponential $\mu_{ij} \geq 0$.
$\mu_{ij} = 0$ means server from station $j$
cannot serve class-$i$ customers.

Now introduce a LLN (= large-deviations) scaling, and
a risk-sensitive cost
Scaled parallel server model

The “size” of the system is $n \in \mathbb{N}$, $n \to \infty$.
$I$ customer classes, $J$ service stations,
$N_j(n) \geq 1$ identical servers in station $j \in \mathcal{J} := \{1, 2, \ldots, J\}$.
$N_j = N_j(n)$ may increase (sub-linearly) with $n$.
Total number of servers $N(n) = \sum_j N_j(n)$, $\lim_{n \to \infty} N(n)/n = 0$.

Arrivals $E_i$, $i \in \mathcal{I} := \{1, 2, \ldots, I\}$ are Poisson $\lambda_i(n) = \lambda_i n$
Service of customer $i$ by server $j$ is exponential $\mu_{ij}(n) \geq 0$ and
    total processing capacity is $N_j(n)\mu_{ij}(n) = n \cdot \mu_{ij}$.
Thus arrival rates and per-station total service rate scale like $n$.
$\mu_{ij}(n) = 0$ means server from $j$ cannot serve class-$i$ customers.
I. Introduction and Motivation - the model

**Scaled parallel server model**

$I$ customer classes, $J$ service stations, arrival rates $\lambda_i(n)$, service rate $\mu_{ij}(n)$.

We control service allocation. An *allocation matrix* is in

$$U := \left\{ u \in \mathbb{R}_+^{I \times J} : \sum_{i \in I} u_{ij} \leq 1, j \in J \right\}. \quad (1)$$

$N_j(n) u_{ij}$ is the number of servers from station $j$ allocated to serve class-$i$ customers (not required to be integer - a server may work on more than one job.)
I. Introduction and Motivation - the model

Processes and cost

Denote by \( \Xi^n = (\Xi^n_i)_{i \in I} \) the number of class-\( i \) customers at \( t \). The scaled version is \( X^n_t = n^{-1} \Xi^n_t \), \( t \geq 0 \). The control process is \( U^n := \{ U^n_{ij} \} \).

The cost is defined through two Lipschitz, monotone increasing (componentwise) functions \( h, g \) where \( h \) is bounded. Each pair \( S = (U^n, X^n) \) induces a probability, so we can define (for a given \( T > 0 \)) a cost-to-go

\[
C^n(t, x, S) = \frac{1}{n} \log \mathbb{E} \left[ e^n \left[ \int_t^T h(X^n_s) ds + g(X^n_T) \right] \right].
\]  

(2)

The value function is

\[
V^n(t, x) = \inf_S C^n(t, x, S), \quad t \in [0, T), \ x \in G^n.
\]  

(3)
Implications of scaling and cost

\[ C^n(t, x, S) = \frac{1}{n} \log E \left[ e^{n \int_t^T h(X^*_s) ds + g(X^*_T)} \right]. \]

Since \( h, g \) are increasing, we penalize large queues.
Since \( n \to \infty \), only the largest values of the queues count, even if those occur with small probability.
But events with small probability occur (large deviations theory) by a change in the process statistics \((\lambda, \mu)\).
We thus need to control “against nature,” which introduces a change in statistics. These changes come “with a price”.
Thus we play a game against nature, and the cost function is a modification of our cost.
Large Deviations

Let $A$ be a set of paths of the scaled queue process over $[0, T]$. There are functions $\ell^U, I^U(A)$ so that

$$P \{ X^n \in A \} \approx e^{-nl(A)}, \quad I(A) = \inf \left\{ \int_0^T \ell(\dot{\phi}(t)) \, dt : \phi \in A \right\}$$

Varadhan’s lemma = Laplace Principle states, for “each” $U$

$$\frac{1}{n} \log \mathbb{E} \left[ e^n \left[ \int_0^T h(X^n_s) \, ds \right] \right] \to \sup_{\phi} \left[ \int_0^T h(\phi(s)) \, ds - \int_0^T \ell^U(\dot{\phi}(s)) \, ds \right].$$

Minimizing over $U$ (process on left, function on right)

$$\inf_U \frac{1}{n} \log \mathbb{E} \left[ e^n \left[ \int_0^T h(X^n_s) \, ds \right] \right] \approx \inf_U \sup_{\phi} \left[ \int_0^T h(\phi(s)) \, ds - \int_0^T \ell^U(\dot{\phi}(s)) \, ds \right]$$

which looks like a differential game.
III. Results

The limit problem

We relate the limit of $V^n$, as $n \to \infty$, to a PDE of Hamilton-Jacobi-Isaacs type.

**Theorem**

Given $t \in [0, T]$ and $x^n \to x$,

$$\lim_{n \to \infty} V^n(t, x^n) = V(t, x),$$

where $V$ is the unique viscosity solution of (7) below.

Below we shall provide an additional interpretation of the limit $V$. 
Notation: set $m_0 = \left( (\lambda_i)_{i \in I}, (\mu_{ij})_{i \in I, j \in J} \right) \in M := \mathbb{R}_+^I \times \mathbb{R}_+^{I \times J}$.

Denote members of $M$ as $m = \left( (\bar{\lambda}_i)_{i \in I}, (\bar{\mu}_{ij})_{i \in I, j \in J} \right)$.

A member $m \neq m_0$ of $M$ is a perturbed set of parameters.

Let $l(x) = x \log x - x + 1$ if $x \geq 0$ and $= \infty$ otherwise.

For $u \in U$ and $m \in M$, let

$$\nu(u, m) = \sum_i \bar{\lambda}_i e_i - \sum_{ij} u_{ij} \bar{\mu}_{ij} e_i$$  \hspace{1cm} (4)

$$\rho(u, m) = \sum_i \lambda_i l\left( \frac{\bar{\lambda}_i}{\lambda_i} \right) + \sum_{ij} u_{ij} \mu_{ij} l\left( \frac{\bar{\mu}_{ij}}{\mu_{ij}} \right),$$ \hspace{1cm} (5)

$\nu$ is the “mean drift” of the perturbed system under the control $u$

$\rho$ is the “cost” of perturbing the system from the original parameter values.
Let $G$ denote the positive quadrant and

$$H(p) = \inf_{u \in U} \sup_{m \in M} \left[ \langle p, v(u, m) \rangle - \rho(u, m) \right], \quad p \in \mathbb{R}^I. \quad (6)$$

Let $I(x) = \{ i \in \mathcal{I} : x_i = 0 \}$ denote indices of “empty queues.”

Define the HJI equation ($V_t$ is the derivative of $V$ w.r.t. $t$, and $DV$ the gradient of $V$ w.r.t. $x$):

$$\begin{cases} V_t + H(DV) + h = 0 & \text{in } [0, T) \times G^o, \\ \langle DV(t, x), e_i \rangle = 0 & x \in \partial G, i \in I(x), \quad (7) \\ V(T, x) = g(x) & x \in G. \end{cases}$$

A solution to equation (7) is in the viscosity sense.
The game

Γ denotes the Skorohod map: prevents paths from exiting the positive quadrant.

Denote by $u$ a path of controls, $m$ a path of parameter values. $v$ is the speed. The differential game has dynamics

$$
\psi(s) := \int_t^s v(u(r), m(r)) \, dr
$$

and $\phi = \Gamma(\psi)$. The cost is

$$
c(t, x, u, m) = \int_t^T \left[ h(\phi(s)) - \rho(u(s), m(s)) \right] \, ds + g(\phi(T))
$$
Queues, game and PDE

The game has a value if minimizing over controls and then maximizing of parameters is the same as the reverse order.

Theorem

(i) The game has a value.
(ii) This value is the unique solution of the HJI equation (7).
(iii) The limits of the optimal costs of the stochastic problem equal the value of the game. Namely $x^n \rightarrow x$ implies

$$V^n(t, x^n) \rightarrow V(t, x).$$

Caveat: the definition of the game requires extra care and details.
Can we compute?

Consider the simple case $h = 0$ and $g(x) = \sum c_i x_i$. Define

$$\hat{\lambda}_i = \lambda_i (e^{c_i} - 1), \quad \hat{\mu}_{ij} = \mu_{ij} (1 - e^{-c_i})$$

**Theorem**

\[ W \cdot T \leq V \leq W \cdot T + \gamma \cdot x \text{ where} \]

\[
W = \min_u \sum_i \left( \hat{\lambda}_i - \sum_j u_{ij} \hat{\mu}_{ij} \right) +
\]

Moreover, if $J = 1$ it is optimal to prioritize service according to the (larger) values of $\hat{\mu}_i$. 
IV. What next?

Is the limit control good for the pre-limit? (AGS)
Compute for larger classes
Generalize model?
Moderate deviations (Atar, Biswas)